

Success in Contests¹

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Abstract

The model of two contestants exerting effort to win a prize is very common and widely used in political economy. The contest success function plays a fundamental role in the theory of contests as does the production function in the theory of the firm, yet little about it has been studied. This paper seeks to remedy that gap.

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1. Introduction

Two themes of Nicholas Yannelis's scientific work are the importance of fundamental results of practical importance and the insistence that they not depend upon special or arbitrary assumptions. So, for example, his work on the existence of competitive equilibrium with large commodity spaces in Yannelis and Zame (1986) does not rest upon arbitrary assumptions about preferences, but it does include the commodity spaces which are important to economists. This paper is about political economy rather than competitive equilibrium, but the analysis and results are in the spirit of Nicholas Yannelis.

The model of two contestants exerting effort to win a prize is a common one - and of particular importance in the political economy of conflict, such as voting or lobbying. A key element of the analysis is the contest success function giving the probability of winning as a function of the effort of the contestants. This function plays as fundamental a role in the theory of contests as does the production function in the theory of the firm, yet little about it has been studied. This paper seeks to remedy that gap.

Assumptions about the contest success function vary. In the all-pay auction the greatest effort wins the prize. The widely used Tullock function supposes that the chance of winning is proportional to effort. A great deal is known about the unique mixed strategy equilibrium in the all-pay auction and a great deal is known about pure strategy equilibria when they exist in the Tullock case. The problem is that equilibrium generally involves mixed strategies and except in the case of the all-pay auction very little is known about the structure of mixed strategy equilibria. Here we address the basic question of when it is that lower cost of effort results in greater success - that is a greater probability of winning or a greater payoff - across equilibria and contest success functions.

In the spirit of Nicholas Yannelis we do not assume particular functional forms. Rather, we allow general contest success functions of the type that are important to economists including the possibility that there is a discontinuous probability of winning when there is a tie, and we allow for general continuous cost functions. We show that Nash equilibria always exist. We take as our measure of success of a contestant her equilibrium utility as a fraction of the prize - that is, how close the contestant is to achieving the goal of winning the prize at no cost.

We observe first that when the contest success function is continuous and costs are high enough, there will be a unique equilibrium in which neither contestant chooses to provide any effort so that lower cost does not provide greater success. More generally, we should be concerned that it might be the case - as it is in the war of attrition - that there can be pre-emptive equilibria in which the higher cost contestant provides a high effort and by doing so discourages the lower cost contestant. Then, we prove three main results. First, there cannot be a pre-emptive equilibrium in which the higher cost contestant has greater success. Second, a contestant with a sufficiently great cost advantage always has greater success. Finally, if the cost advantage is a homogeneous one, then the lower cost contestant always has greater success.

As an application we study contestants representing groups of different sizes competing for a common prize with identical per capita cost technology. We show that when the marginal cost of effort is increasing, the large group has greater success, while when there are decreasing marginal cost of effort the small group has greater success. We further study the robustness of equilibrium by proving a basic upper hemi-continuity result. The underlying mathematics derives from the study of the convergence of monotone functions on rectangles. This enables us to conclude, for example, that contest success functions that converge pointwise to the all-pay auction do not have pure strategy equilibria. More broadly it shows that greater success is a robust property shared by neighboring contest success functions.

A fundamental result of Whitney (1934) enables us to approximate discontinuous contest success functions by real analytic contest success functions. This is important because most functional forms used by economists are real analytic. Remarkably, considering that little is known in general about mixed strategy equilibria in games with a continuum of actions, we establish that when the contest success function is real analytic, the support of mixed strategy equilibria must be finite. Hence, for example, if the contest success function is the normal cumulative distribution applied to the difference in effort levels, and the variance decreases to zero so that the contest success function approaches the all-pay auction, then equilibria have finite support converging weakly in the limit to the continuous uniform distribution that is the unique equilibrium of the all-pay auction.

2. The Model

Two contestants $j \in \{1, -1\}$ compete for a prize worth $V_j > 0$ to contestant j . Each contestant chooses an effort level $e_j \geq 0$. The probability of contestant j winning the prize is given by a *contest success function* $0 \leq p(e_j, e_{-j}) \leq 1$ that depends on the efforts of the two contestants and not on their names.

The contest success function is assumed to be continuous for $e_j \neq e_{-j}$, non-decreasing in e_j , and it must satisfy the symmetry condition $p(e_j, e_{-j}) + p(e_{-j}, e_j) = 1$. Note that we allow for a discontinuous upward jump in the winning probability when we move away from $e_j = e_{-j}$, but require that when there is a tie the probability of winning is $1/2$. Two standard contest resolution functions have this type of discontinuity: the all-pay auction in which the highest effort wins for sure and the Tullock function where the probability of winning is given by $e_j/(e_j + e_{-j})$ which is discontinuous when there is a tie at zero.

The cost of effort e_j is $V_j c_j(e_j)$ and it is incurred regardless of the outcome of the contest. The function $c_j(\cdot)$ measures cost relative to the value of the prize $V_j > 0$. We assume that $c_j(\cdot)$ is continuous, non-decreasing, it satisfies $c_j(0) = 0$, and for some w_j called the *willingness to bid* $c_j(w_j) = 1$. To avoid degeneracy we assume that for contestant -1 the cost function $c_{-1}(\cdot)$ is strictly increasing at the origin.

Since choosing effort higher than the willingness to bid is strictly dominated by choosing zero effort, we may restrict the choice of effort to $[0, W]$,

where $W \geq \max\{w_j, w_{-j}\}$. Hence, a strategy for contestant j is a cdf F_j on $[0, W]$. Define $p(F_j, F_{-j}) \equiv \int_0^W \int_0^W p(e_j, e_{-j}) dF_j(e_j) dF_{-j}(e_{-j})$ and $c_j(F_j) \equiv \int_0^W c_j(e_j) dF_j(e_j)$. A *Nash equilibrium* is a pair of strategies (F_j, F_{-j}) such that for each contestant j and all strategies \tilde{F}_j we have

$$p(F_j, F_{-j}) - c_j(F_j) \geq p(\tilde{F}_j, F_{-j}) - c_j(F_j).$$

Since this is an expected utility model this definition is equivalent to restricting deviations to pure strategies e_j .

Existence of Equilibrium

As we allow for discontinuities in $p(e_j, e_{-j})$ along the diagonal where $e_j = e_{-j}$ it is useful to have a measure of the discontinuity.

Define

$$p^+(e) = \lim_{\epsilon \rightarrow 0} \sup_{k \in \{1, -1\}, |e_k - e| \leq \epsilon} p(e_1, e_{-1}).$$

This is the greatest chance of winning for any effort pair (e_j, e_{-j}) close to (e, e) . In particular $p^+(e) - 1/2$ is zero at a point of continuity and positive at a point of discontinuity. There is a simple way of computing $p^+(e)$:

Lemma 1. $p^+(e) = \lim_{\epsilon \rightarrow 0^+} p(e + \epsilon, e)$.

Proof. By definition $p^+(e) \geq \lim_{\epsilon \rightarrow 0^+} p(e + \epsilon, e)$. We show that in fact $p^+(e) \leq p(e + \epsilon, e)$ so that the opposite inequality holds as well. Let $e_k^n \rightarrow e$ be such that $p(e_k^n, e_{-j}^n) \rightarrow p^+(e)$. Fix $e + \epsilon$. For n sufficiently large $e_k^n < e + \epsilon$. Hence $p(e + \epsilon, e_{-j}^n) \geq p(e_k^n, e_{-j}^n)$. Consider the limit as $n \rightarrow \infty$. Since $p(e + \epsilon, e)$ is a point of continuity of $p(e_j, e_{-j})$ we have $\lim p(e + \epsilon, e_{-j}^n) = p(e + \epsilon, e)$ while by definition $p(e_k^n, e_{-j}^n) \rightarrow p^+(e)$. Hence $p^+(e) \leq p(e + \epsilon, e)$. \square

Lemma 2. *If F_{-j} has an atom at e and $p(e, e)$ is a point of discontinuity then $e_j = e$ is not a best-response by j to F_j .*

Proof. The idea is that it would be better to choose a little bit more effort than e so as to break the tie and get a jump in the probability of winning at trivial additional cost. Specifically, suppose that $-j$ has an atom $f_{-j}(e)$ at e . If j provides effort $e + \epsilon$ instead of e then j gains at least

$$f_{-j}(e)(p^+(e) - 1/2) + c(e) - c(e + \epsilon).$$

In the limit as $\epsilon \rightarrow 0$ this is strictly positive proving the result. \square

Theorem 1. *A Nash equilibrium exists and in every Nash equilibrium the probability of a tie at a point of discontinuity is zero.*

We prove existence in Corollary 3 below. Notice that if $p(e, e)$ is a point of discontinuity then by Lemma 2 both contestants cannot have an atom at e so the probability of (e, e) is zero, which implies:

Corollary 1. *If $p^+(e) > 0$ for all $0 \leq e \leq W$ and if both contestants have the same costs there is no symmetric pure strategy equilibrium.*

In other words: the basic property that the all-pay auction has no symmetric pure strategy equilibrium holds for any contest success function that is discontinuous along the diagonal.

3. Cost and Success

We are interested in the case in which 1 has a cost advantage. Our goal is to analyze the extent to which this translates to greater success in the contest. One measure of success is a greater probability of winning: we say that j has *outcome success* if $p(F_j, F_{-j}) > 1/2$ or equivalently $p(F_j, F_{-j}) > p(F_{-j}, F_j)$. This, however, fails to take into account the cost of the resources used in achieving success, so we say that j has *greater success* if $p(F_j, F_{-j}) - c_j(F_j) > p(F_{-j}, F_j) - c_j(F_{-j})$, that is, j gets a greater fraction of achievable utility than $-j$.

The simplest notion of cost advantage is that of a *pure cost advantage*: here $e > 0$ results in $c_1(e) < c_{-1}(e)$. For example in the case of the all-pay auction where $e_j > e_{-j}$ results in $p(e_j, e_{-j}) = 1$ we have that

Theorem 2. *In the all-pay auction if 1 has a pure cost advantage then 1 has greater success.*

Proof. Define $\bar{e}_{-1} \equiv \max \text{supp} F_{-1}$. Consider the strategy for 1 of providing effort $e_\epsilon \equiv \bar{e}_{-1} + \epsilon$. In the all-pay auction this guarantees a win, so

$$p(F_1, F_{-1}) - c_1(F_1) \geq 1 - c_1(e_\epsilon).$$

By the continuity of c_1 this implies

$$p(F_1, F_{-1}) - c_1(F_1) \geq 1 - c_1(\bar{e}_{-1}).$$

Because 1 has a pure cost advantage, the right hand side of the inequality is strictly larger than $1 - c_{-1}(\bar{e}_{-1})$. Because $\bar{e}(F_{-1}) \in \text{supp} F_{-1}$ there is a sequence $e^n \rightarrow \bar{e}_{-1}$ with

$$p(e^n, F_1) - c_{-1}(e^n) = p(F_{-1}, F_1) - c_{-1}(F_{-1}).$$

By the continuity of c_{-1} this implies

$$1 - c_{-1}(\bar{e}_{-1}) \geq p(F_{-1}, F_1) - c_{-1}(F_{-1}).$$

Hence it is indeed the case that 1 is more successful. \square

Our goal is to understand how this result extends to the general case. First of all, however, we want to rule out uninteresting cases where the result of Theorem 2 trivially does not extend.

4. Peaceful Equilibria

Consider the following example.

Example 1. Suppose that $c_1(e_1) = e_1, c_{-1}(e_{-1}) = 2e_{-1}$ so that 1 has a pure cost advantage but that $p(e_j, e_{-j}) \equiv 1/2$ so that effort does not matter. The the unique equilibrium is for each to provide zero effort so both get 1/2 and neither is more successful.

We define *peaceful* equilibria those in which both contestants choose to provide zero effort and have a probability of winning and utility equal to 1/2.⁴ To have a *contested* equilibrium in which this is not the case we must rule out situations such as Example 1 in which the cost function rises too fast relative to the steepness of the contest success function. We begin with the relevant definitions.

We say that contestant j has *very high cost* if for all $e_j > 0$ we have $c_j(e_j) > \sup_{e_{-j}} p(e_j, e_{-j}) - p(0, e_{-j})$. Contestant j has *high cost* if for all $e_j > 0$ we have $c_j(e_j) > p(e_j, 0) - p(0, 0)$. Since $\sup_{e_{-j}} p(e_j, e_{-j}) - p(0, e_{-j}) \geq p(e_j, 0) - p(0, 0)$ very high cost implies high cost. Notice that when p is discontinuous at $(0, 0)$ as it is in the all-pay auction or the Tullock case, high cost is ruled out because $c_j(e_j)$ is continuous and $c_j(0) = 0$. By contrast, we say that contestant j has *low cost* if for some e_j we have $c_j(e_j) < p(e_j, 0) - p(0, 0)$, and in particular high cost and low cost are mutually exclusive.

Theorem 3. *If 1 has high cost and -1 has very high cost then the unique equilibrium is peaceful and neither provides effort. If both have high cost there is a peaceful equilibrium in which neither provides effort. If 1 has low cost all equilibria are contested.*

Proof. The condition for very high cost for -1 may be written as $p(e_{-1}, e_1) - c_{-1}(e_{-1}) < p(0, e_{-1}) - c_{-1}(0)$ for all $e_{-1} > 0$ so that it is strictly dominant to provide zero effort. The condition for 1 to have high cost may be written as $p(0, 0) - c_1(0) > p(e_1, 0) - c_1(e_1)$ for all $e_1 > 0$ so it is strictly optimal for 1 to provide zero effort as well. Similarly if both have high cost then each finds it optimal to provide zero effort when the other is doing so. Finally, at a peaceful equilibrium since $c_{-1}(e_{-1})$ is assumed to be strictly increasing at the origin, as we noted above, it must be that -1 provides zero effort. The condition for 1 having low cost may be written as $p(e_1, 0) - c_1(e_1) > p(0, 0) - c_1(0)$ implying that 1 gets strictly more than 1/2 in equilibrium. This requires that the chance of 1 winning is greater than 1/2 contradicting the definition of a peaceful equilibrium. \square

⁴Notice that we have assumed that cost for -1 is strictly increasing at the origin, so in a peaceful equilibrium -1 must provide zero effort.

5. Contested Equilibria

We now focus on contested equilibria and we first show that the notion of pure cost advantage must be strengthened if it is to apply to cases with contested equilibria.

Example 2. Here we construct a contested pure strategy equilibrium in which 1 has a pure cost advantage but -1 has greater success. Take $p(e_j, e_{-j}) = (1/2) + (1/2)(e_j - e_{-j})$ truncated by 0 below and 1 above. The cost function for 1 is $c_1(e_1) = (4/7)(e_1 - 1)$ for $e_1 \geq 1$ and 0 otherwise. For -1 it is $c_{-1}(e_{-1}) = (3/7)e_{-1}$ for $0 \leq e_{-1} \leq 2$ and $6/7 + (4/7)(e_{-1} - 2)$ otherwise. At $e = 0$ we have $c_1 = c_{-1} = 0$. At $e = 1$ we have $c_1 = 0$, and $c_{-1} = 3/7$. At $e = 2$ we have $c_1 = 4/7$, and $c_{-1} = 6/7$. Above 2 the cost difference remain equal to $2/7$ in favor of -1 . So 1 has a pure cost advantage. We claim that $(e_1, e_{-1}) = (1, 2)$ is a pure strategy equilibrium. Here 1 loses for certain and has no cost so gets 0 while -1 wins for sure and has a cost of $6/7$ so gets $1/7$. Hence certainly -1 is more successful. To see this is an equilibrium observe that 1 is indifferent to reducing effort below 1: there is no cost and no chance of winning there. Increasing effort above 1 increases the chances of winning at the rate of $1/2$ while it increases costs at the rate of $4/7$ so in fact $e_1 = 1$ is optimal for contestant 1. For -1 reducing effort below 2 reduces the chances of winning at the rate of $1/2$ but decreases costs only at the rate of $3/7$. Increasing effort above 1 has no effect on the chances of winning but simply increases costs. Hence $e_{-1} = 2$ is optimal for contestant -1 .

We introduce two strengthened notions of cost advantage

1. 1 has a *marginal cost advantage* if for $e_2 > e_1$ we have $c_1(e_2) - c_1(e_1) < c_{-1}(e_2) - c_{-1}(e_1)$
2. 1 has a *homogeneous cost advantage* if $c_1(e) = \alpha c_{-1}(e)$ for some $0 < \alpha < 1$

Given these notions, we have that homogeneous cost advantage implies marginal cost advantage, and marginal cost advantage implies pure cost advantage. An important special case of homogeneous cost advantage occurs when both contestants have the same absolute cost: for all e we have $V_1 c_1(e) = V_{-1} c_{-1}(e)$. In this case 1 has a homogeneous cost advantage if and only if the prize is valued more highly: $V_1 > V_{-1}$.

We also generalize the notion of pure strategy equilibrium. We say that F_1, F_{-1} is a *preemptive equilibrium* if either one distribution first order stochastically dominates the other or the two are equal. Equipped with these new definitions we can state our first main result:

Theorem 4. *In a contested equilibrium 1 has greater success if either of the following two conditions is satisfied:*

- (i) 1 has a *marginal cost advantage* and the equilibrium is preemptive,
- (ii) 1 has a *homogeneous cost advantage*.⁵

⁵The proofs of Theorem 4 and of the following Theorem 5 follow from Lemma 3, which we will state and prove at the end of this section.

Notice that in Example 2 while 1 had a pure cost advantage in the range $[1, 2]$, 1 also had higher marginal cost than -1 . This possibility is ruled out by marginal cost advantage. With this assumption 1 has greater success in all preemptive equilibria. For pure strategies this trivially “works” since all pure strategy equilibria are preemptive. Unfortunately pure strategy equilibria do not always exist and we do not have general results about when equilibria are preemptive. If we further strengthen the cost advantage assumption to homogeneous cost advantage then we get a general result for all equilibria pure or mixed.

The following special case of Theorem 4 is useful in a variety of applications.

Corollary 2. *In a contested equilibrium 1 has greater success if either of the following two conditions is satisfied:*

- (i) *Cost is linear and 1 has a pure cost advantage.*⁶
- (ii) *1 has a marginal cost advantage and one contestant provides no effort.*

The Role of Contest Success

Pure cost advantage is defined independent of the contest success function. An alternative approach is to relate the size of the cost advantage to the steepness of the contest success function. When the contest success function is very steep, as in the all pay auction, intuitively we expect that very little cost advantage is needed.

A simple but quite strong form of cost advantage is the following: we say that 1 has a *strong cost advantage over* -1 if for some $e_1 > w_{-1}$, where w_j is the willingness to bid defined earlier, we have $c_1(e_1) < p(e_1, w_{-1}) - p(w_{-1}, w_{-1})$. For example, if 1 has low cost and -1 has high cost then 1 has a strong cost advantage. To understand this condition better fix w_{-1} , -1 's willingness to bid. If contest success has a strict increase above this point, a sufficiently low cost for 1 will always have a strong cost advantage. On the other hand, strong cost advantage in the all-pay auction requires that $c_1(w_{-1}) < 1/2$, while we know that greater success requires only that $c_1(w_{-1}) < 1$.

For this reason we introduce a weaker condition applied over a broader range of effort levels. We say that 1 has a *uniform cost advantage over* -1 if for any $0 \leq e_{-1} \leq w_{-1}$ there is an $e_1 > e_{-1}$ with $c_1(e_1) < c_{-1}(e_{-1}) - (p(e_{-1}, 0) - p(e_1, e_{-1}))$. Notice that this condition is satisfied in the all-pay auction provided that 1 has a cost advantage. It is also satisfied in a difference model in which $p(e_1, e_{-1}) = p(e_1 - e_{-1}, 0)$ if $c_1(2e_1) < c_{-1}(e_1)$. One particularly important case of a uniform cost advantage arises when there is a common underlying strictly increasing cost function $c_2(e)$ but contestant 1 has an effort advantage of $\bar{e}_1 > 0$, meaning that the probability that 1 wins with underlying effort \tilde{e}_1 is given by $p(\tilde{e}_1 + \bar{e}_1, e_{-1})$. This can be made equivalent to the original model by defining $c_1(e_1) = \tilde{c}_2(e_1 - \bar{e}_1)$ for $e_1 \geq \bar{e}_1$ and 0 otherwise. Notice that in this case the cost advantage cannot be homogeneous. Equipped with these new definitions we have two

⁶This assumption is very popular in the literature.

additional sufficient conditions for greater success in a contested equilibrium to those of Theorem 4:

Theorem 5. *In a contested equilibrium 1 has greater success if either of the following two conditions is satisfied:*

- (iii) *1 has a strong cost advantage,*
- (iv) *1 has a uniform cost advantage.*

Application to Group Size

We now consider that the contestants are groups denoted by $j = L, S$ for large and small and we want to determine which is the advantaged group. We specialize to the case where the value of the prize is the same for both groups $V_L = V_S = V$. This corresponds to a situation where, for example, there are a fixed number of political jobs or money that accrues to the winner who divides them among the group and may be thought of as prototype of a political contest involving transfer payments. We denote by $\eta_j > 0$ the size of group j where $\eta_S + \eta_L = 1$ and we assume that both groups have the same per capita cost as a function of per capita effort, and examine the implications for success by applying Theorem 4.

Specifically, if per capita effort is e_j/η_j , the common per capita cost is denoted by $C(e_j/\eta_j)$ and it is assumed differentiable, at least for large enough effort levels. Cost for group j is then given by $c_j(e_j) = \eta_j C(e_j/\eta_j)$. We say that per capita cost is *asymptotically convex* if $\lim_{x \rightarrow \infty} C'(x) = \infty$ and *asymptotically concave* if $\lim_{x \rightarrow \infty} C'(x) = 0$. Recalling our definition that a group has greater success if it gets a greater fraction of achievable utility than the other group, we have that:

Theorem 6. *If per capita cost is asymptotically convex and for some $\hat{e} > 0$ and $\lambda > 1$ we have $\lambda C(\hat{e}) \leq p(\hat{e}, 0) - 1/2$, then a large enough group has greater success; if per capita cost is asymptotically concave and p is strictly increasing then a small enough group has greater success.*

Proof. If per capita cost is asymptotically convex we can calculate w_S from $\eta_S C(w_S/\eta_S) = 1$. As $\eta_L \rightarrow 1$ and so $\eta_S \rightarrow 0$ we see that we must have $C(w_S/\eta_S) \rightarrow \infty$. We can then compute $\eta_S C(w_S/\eta_S)$ using L'Hospital's rule: $\eta_S C(w_S/\eta_S) \rightarrow w_S C'(w_S/\eta_S)$. Since $C(w_S/\eta_S) \rightarrow \infty$ so does w_S/η_S so we must have $w_S \rightarrow 0$. For η_L sufficiently large we have $c_L(\hat{e}) = \eta_L C(\hat{e}/\eta_L) \leq C(\hat{e})$. Hence $c_L(\hat{e}) < \lambda c_L(\hat{e}) \leq p(\hat{e}, 0) - 1/2$. For $w_S < \hat{e}$ we have $p(\hat{e}, w_S)$ continuous so for sufficiently small w_S we have $c_L(\hat{e}) < p(\hat{e}, w_S) - 1/2$. Hence L has a strong cost advantage so greater success.

If per capita cost is asymptotically concave let $C(\hat{w}) = 1$. As $\eta_L \rightarrow 1$ we see that $w_L \rightarrow \hat{w}$. Since p is strictly increasing, we can find a $\hat{e} > \hat{w}$ so that $p(\hat{e}, \hat{w}) - 1/2 > 0$. For w_L sufficiently close to \hat{w} we have p continuous, so also $p(\hat{e}, w_L) - 1/2 > \epsilon > 0$. Consider $c_S(\hat{e}) = \eta_S C(\hat{e}/\eta_S)$. If C is bounded this goes to zero. If it is unbounded above from L'Hospital's rule it goes to $\hat{e} C'(\hat{e}/\eta_S)$ which goes to zero by asymptotic concavity. Hence for sufficiently small η_S we have $c_S(\hat{e}) < \epsilon$ so that S has a strong cost advantage so greater success. \square

We now analyze more closely the example $C(x) = \xi x^\alpha$. Here we have

$$c_j(e) = \xi \eta_j (e/\eta_j)^\alpha = \xi \eta_j^{1-\alpha} e^\alpha = (\eta_j/\eta_{-j})^{1-\alpha} c_{-j}(e).$$

Hence if $\alpha > 1$ so that $C(x)$ is convex (and asymptotically convex) then L has a uniform cost advantage, while if $\alpha < 1$ so that $C(x)$ is concave (and asymptotically concave) then S has a uniform cost advantage.

Take first the convex case of $\alpha > 1$. Since $c_j(e)$ has zero slope at the origin the low cost assumption is satisfied by both so L has low cost hence greater success.

In the concave case we must check that the small group S has low cost, otherwise there will be a zero effort equilibrium in which neither group has greater success. We need $\xi \eta_S^{1-\alpha} e^\alpha < p(e, 0) - 1/2$ for some e , or $\xi < \eta_S^{\alpha-1} e^{-\alpha} (p(e, 0) - 1/2)$. Notice that for η_S sufficiently small this is satisfied for any increasing p which is why no extra condition was needed in Theorem 6. Here a sufficient condition is

$$\xi < 2^{1-\alpha} \max_e e^{-\alpha} (p(e, 0) - 1/2),$$

in which case S has greater success.

Proof of the Main Theorem

Theorems 4 and 5 follow from (i),(ii), (iii), and (iv) below:

Lemma 3. *In a contested equilibrium 1 has greater success if any of the following conditions are satisfied*

- (0) *she has a pure cost advantage and -1 does not have outcome success,*
- (i) *she has a marginal cost advantage and the equilibrium is preemptive,*
- (ii) *she has a homogeneous cost advantage,*
- (iii) *she has a strong cost advantage,*
- (iv) *she has a uniform cost advantage.*

Proof. From optimality of F_j and symmetry we have

$$p(F_j, F_{-j}) - c_j(F_j) \geq p(F_{-j}, F_{-j}) - c_j(F_{-j}) = 1/2 - c_j(F_{-j}). \quad (1)$$

By subtraction we also have

$$p(F_j, F_{-j}) - 1/2 \geq c_j(F_j) - c_j(F_{-j}). \quad (2)$$

First, we show (0). Suppose that 1 has a pure cost advantage but does not have greater success. Then from equation 1

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) \geq p(F_1, F_{-1}) - c_1(F_1) \geq 1/2 - c_1(F_{-1}). \quad (3)$$

If -1 is not providing effort this implies $p(F_{-1}, F_1) \geq 1/2$ and p non-decreasing implies $p(F_{-1}, F_1) \leq 1/2$ so $p(F_{-1}, F_1) = 1/2$. Since this is not a peaceful equilibrium it must be that $c_1(F_1) > 0$ so $p(F_1, F_{-1}) - c_1(F_1) = 1/2 - c_1(F_1) <$

1/2 while choosing $e_1 = 0$ gives a utility of 1/2 contradicting the fact that 1 is playing optimally. If -1 is providing effort by the pure cost advantage equation

$$1/2 - c_1(F_{-1}) > 1/2 - c_{-1}(F_{-1})$$

From equation 3 this gives $p(F_{-1}, F_1) > 1/2$. Consequently -1 has outcome success. This proves (0).

To show (i), notice that from equation 2 with $j = 1$ we have

$$p(F_1, F_{-1}) - 1/2 \geq c_1(F_1) - c_1(F_{-1}).$$

From symmetry this gives

$$-p(F_{-1}, F_1) + 1/2 \geq c_1(F_1) - c_1(F_{-1})$$

or

$$p(F_{-1}, F_1) - 1/2 \leq c_1(F_{-1}) - c_j(F_1). \quad (4)$$

From equation 2 with $j = -1$ we have

$$p(F_{-1}, F_1) - 1/2 \geq c_{-1}(F_{-1}) - c_{-1}(F_1)$$

Hence

$$c_1(F_{-1}) - c_1(F_1) \geq c_{-1}(F_{-1}) - c_{-1}(F_1). \quad (5)$$

Suppose that 1 has a marginal cost advantage. If F_1 first order stochastically dominates F_{-1} or the two are equal then -1 does not have a outcome advantage so 1 has greater success by (0). Suppose instead that F_{-1} first order stochastically dominates F_1 . For $e_2 > e_1$ the condition for marginal cost advantage can be written as $c_{-1}(e_2) - c_1(e_2) > c_{-1}(e_1) - c_1(e_1)$. It follows that $c_{-1}(F_{-1}) - c_1(F_{-1}) > c_{-1}(F_1) - c_1(F_1)$. This contradicts equation 5. This shows (i).

Next, we show (ii). Suppose that 1 has a homogeneous cost advantage. From equation 5

$$c_1(F_{-1}) - c_1(F_1) \geq c_{-1}(F_{-1}) - c_{-1}(F_1) = (1/\alpha)(c_1(F_{-1}) - c_1(F_1)).$$

Since $\alpha < 1$ it follows that $c_1(F_{-1}) - c_1(F_1) \leq 0$. From equation 4

$$p(F_{-1}, F_1) - 1/2 \leq c_1(F_{-1}) - c_1(F_1) \leq 0$$

so -1 does not have an outcome success. If 1 does not have an outcome success either, then, it must be that $p(F_{-1}, F_1) = 1/2$ so that also $p(F_1, F_{-1}) = 1/2$. By (0) we may assume that -1 does not provide zero effort with probability one so by cost advantage

$$p(F_1, F_{-1}) - c_1(F_{-1}) > p(F_1, F_{-1}) - c_{-1}(F_{-1}) = p(F_{-1}, F_1) - c_{-1}(F_{-1})$$

and indeed 1 instead has greater success. If 1 does have outcome success by (0) 1 has greater success. This proves (ii).

We now show (iii). If 1 has a strong cost advantage then there is a \hat{e}_1 with $c_1(\hat{e}_1) < p(\hat{e}_1, w_{-j}) - p(w_{-j}, w_{-j}) = p(\hat{e}_1, w_{-j}) - 1/2$. Hence $p(\hat{e}_1, w_{-j}) - c_1(\hat{e}_1) > 1/2$. Observe that $F_{-1} \leq w_{-1}$ so $p(\hat{e}_1, w_{-j}) \leq p(\hat{e}_1, F_{-1})$. Finally, from optimality

$$p(F_1, F_{-1}) - c(F_1) \geq p(\hat{e}_1, F_{-1}) - c(\hat{e}_1) \geq p(\hat{e}_1, w_{-j}) - c_1(\hat{e}_1) > 1/2$$

which as both contestants cannot have a utility greater than 1/2 implies greater success. This proves (iii)

Finally we prove (iv). Let \hat{e}_{-1} be the top of the support of the equilibrium F_{-1} . Let $e_{-1}^n \leq \hat{e}_{-1}$ with $e_{-1}^n \rightarrow \hat{e}_{-1}$ and $p(e_{-1}^n, F_1) - c_{-1}(e_{-1}^n) = p(F_{-1}, F_1) - c_{-1}(F_{-1})$. Since at points of discontinuity of p the jump is up this implies

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) \leq p(\hat{e}_{-1}, 0) - c_{-1}(\hat{e}_{-1}).$$

From the definition of a uniform cost advantage there is a \hat{e}_1 such that

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) < p(\hat{e}_1, \hat{e}_{-1}) - c_1(\hat{e}_1).$$

Moreover because \hat{e}_1 is the top of the support of F_{-1} we get

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) < p(\hat{e}_1, F_{-1}) - c_1(\hat{e}_1)$$

By optimality of F_1 this gives

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) < p(F_1, F_{-1}) - c_1(F_1)$$

that is to say, greater success. \square

6. Existence and Robustness

In order to prove existence we will now deal with sequences of contests $p_n(e_1, e_{-1}), c_{1n}(e_1), c_{-1n}(e_{-1})$. To make sense of this, we now give a slightly more formal definition of a contest. A *contest* on W is a contest success function $p(e_j, e_{-j}) \geq 0$ for $0 \leq e_1, e_{-1} \leq W$, which is non-decreasing in the first argument, continuous except possibly at $e_j = e_{-j}$, and satisfying the symmetry condition $p(e_j, e_{-j}) + p(e_{-j}, e_j) = 1$ together with a pair of cost functions $c_j(e_j) \geq 0$ non-decreasing and continuous with $c_j(0) = 0$, $c_j(W) > 1$, and c_{-1} strictly increasing at 0. For a contest on W we take the strategy space to be of cumulative distribution functions on $[0, W]$. Theorem 15 in the Appendix shows that

Theorem 7. *Suppose $p_n(e_1, e_{-1}) \rightarrow p_0(e_1, e_{-1}), c_{jn}(e_j) \rightarrow c_{j0}(e_j)$ are a sequence of contests in W and that F_{1n}, F_{-1n} are equilibria for n converging weakly to F_{10}, F_{-10} . Then $p_n(F_{jn}, F_{-jn}) \rightarrow p_0(F_{j0}, F_{-j0}), c_{jn}(F_{jn}) \rightarrow c_{j0}(F_{j0})$ for both j and F_{10}, F_{-10} is an equilibrium for $p_0(e_1, e_{-1}), c_{j0}(e_j)$.*

We should emphasize that this result requires only the pointwise convergence of p_n, c_{jn} . Pointwise convergence is easy to check, but, as shown in the

Appendix, has strong consequences for non-decreasing functions on rectangles. If the limit is continuous the convergence is uniform. Even if the limit is discontinuous on the diagonal - as we allow for contest success function - convergence is uniform on the set of effort pairs that is bounded away from the diagonal.

We say that a contest is *well-behaved* if $p(e_j, e_{-j}) > 0$, p is strictly increasing in the first argument, c_j is strictly increasing, and both have an extension to an open neighborhood of $[0, W] \times [0, W]$ that is real analytic. In Appendix 12 we show

Theorem 8. *If p, c_j is a contest on W then there is a sequence of well-behaved contests p_n, c_{jn} on W with $p_n(e_j, e_{-j}) \rightarrow p(e_j, e_{-j}), c_{jn}(e_j) \rightarrow c_j(e_j)$ for every $(e_1, e_{-1}) \in [0, W] \times [0, W]$.*

Since real analytic functions are continuous, as an immediate corollary we have:

Corollary 3. *Nash equilibria exist. If both contestants have the same costs there is a symmetric Nash equilibrium.*

We are interested in understanding properties of contests that are robust. By a *property* we mean a statement $\Pi(p, c, F)$ such as: there is complete rent dissipation, contestant 1 has greater success, or one contestant has zero utility. We say that a property is true in a contest if it is true for all equilibria of the game. We say that a $\Pi(p, c, F)$ in p, c is *robust* if whenever it is true in p, c then for every sequence p_n, c_n converging pointwise to p, c and for n sufficiently large the property is true in p_n, c_n .

Corollary 4. *Any strict inequality concerning equilibrium utility, probability of winning, or cost is robust.*

Proof. Suppose not. Then there exists a subsequence in which $\Pi(p_n, c_n, F_n)$ is false. Since the space of strategies is compact every subsequence contains a further subsequence that converges weakly to some F . By Theorem 7 F is an equilibrium and utility, winning probability, and cost converge. Hence as the strict inequality is presumed to be satisfied for F for all sufficiently large n it was satisfied for $\Pi(p_n, c_n, F_n)$, a contradiction. \square

Finite Support

In Appendix 14 we show that well-behaved contests have a relatively simple equilibrium structure:

Theorem 9. *Suppose that $c_1(b_j) = 0$ for $0 \leq b_1 \leq \underline{w}_1$ and if $\underline{w}_1 > 0$ we require that $p(b_j, b_{-j})$ is strictly increasing (so in particular in any equilibrium $\lim_{w \uparrow \underline{w}_1} F_1(w) = 0$). Suppose as well that $c_j(W) > 1$. If $p(b_j, b_{-j}), c_j(b_j)$ have real analytic extensions to an open neighborhood of $[\underline{w}_1, W] \times [0, W]$ then every equilibrium has finite support.*

Ewerhart (2015) who developed the technique we use in the appendix analyzed the symmetric Tullock contest for high degrees of sensitivity. That function is not well-behaved since it is discontinuous at zero and without the extension of analyticity below zero the finiteness result fails: in the Tullock case the support is countable with a single accumulation point at zero.

7. Rent Dissipation and Zero Surplus

Historically the literature on contests has focused especially on the idea of *complete rent dissipation*, meaning that both contestant get zero, competing so hard that the gains are cancelled by the costs. This is the case in the symmetric all pay auction. Notice that this is ruled out if one contestant provides zero effort⁷ and by a contested equilibrium in which one contestant has a greater success.

Also important in the literature has been the weaker situation in which one contestant gets nothing - this is the case in every all pay auction, symmetric or not. It turns out that the possibility of a contestant getting nothing is quite exceptional. We say that a property is *generic* if it is robust and if for any p, c_1, c_{-1} for which it is not true there is a sequence p_n, c_{jn} converging pointwise to p, c_j in which it is true.

We formally define properties corresponding to dissipation:

1. *no dissipation*: in equilibrium $c_1(F_1) + c_{-1}(F_{-1}) = 0$
2. *partial dissipation*: in equilibrium $0 < c_1(F_1) + c_{-1}(F_{-1}) < 1$
3. *some dissipation*: in equilibrium $0 < c_1(F_1) + c_{-1}(F_{-1})$
4. *complete dissipation*: in equilibrium $c_1(F_1) + c_{-1}(F_{-1}) = 1$
5. γ -*dissipation*: in equilibrium $c_1(F_1) + c_{-1}(F_{-1}) > \gamma$ where $0 \leq \gamma < 1$

Notice that complete dissipation means γ -dissipation for every $0 \leq \gamma < 1$. Moreover, contested equilibrium implies some dissipation. If in addition one contestant has greater success then there is partial dissipation. We have

Theorem 10. *Concerning rent dissipation:*

- (i) *there is a subset of contests with no dissipation that are robust*
- (ii) *the entire set of contests with some (or partial) dissipation is robust*
- (iii) *contests without complete dissipation are generic*
- (iv) *contests with γ -dissipation are robust*

Proof. (i) The property of very high cost for j is $c_j(e_j) > \sup_{e_{-j}} p(e_j, e_{-j}) - p(0, e_{-j})$ which is robust by Corollary 4. Suppose that $p(e_j, e_{-j})$ is continuous. In this case the property of high cost $c_j(e_j) > p(e_j, 0) - p(0, 0)$ for all $e_j > 0$ is equivalent to $\max_{e_j} c_j(e_j) - p(e_j, 0) - p(0, 0) > 0$, also robust by Corollary 4. It follows that for continuous $p(e_j, e_{-j})$ the property of 1 having high cost and -1 having very high cost is robust. By Theorem 3 the latter two conditions imply a unique peaceful equilibrium and hence no dissipation.

⁷The other one cannot get less than $1/2$ as this is obtainable by providing zero effort.

Part (ii) follows directly from Corollary 4 and the fact that some (partial) dissipation is defined by a strict cost inequality

For (iii) we show the slightly stronger result that both contestants getting positive utility is generic. Strict inequality concerning utility is robust by Corollary 4: this proves that both contestants getting positive utility is robust. We will show that for any p_0, c_{j0} there is a sequence p_n, c_{jn} converging uniformly to p_0, c_{j0} in which each contestant gets positive utility in every equilibrium, and this will complete the proof.

For costs we take $c_{jn} = c_{j0}$. Then take $1 > \lambda_n > 0$ to be a sequence converging to zero and define

$$p_n(e_j, e_{-j}) = (1 - \lambda_n)p_0(e_j, e_{-j}) + \lambda_n\Phi(e_j - e_{-j})$$

where Φ is the standard normal cumulative distribution function. This obviously converges uniformly to $p_0(e_j, e_{-j})$. Moreover, for $0 \leq e_j \leq W$ we have $p_n(e_j, e_{-j}) \geq \lambda_n\Phi(-W)$. Hence bidding zero gets at least $\lambda_n\Phi(-W) > 0$ so this is obtained by both contestants in any equilibrium.

The proof of (iv) follows from taking an anomalous subsequence and then finding one on which F_n converges. \square

Notice that (iii) states that complete dissipation is not robust and (iv) that contests near those with complete dissipation - so for example close to symmetric all pay - have nearly complete dissipation.

8. Resource Limits

Some of the existing contest models truncate the effort level above: for example, there might be only a limited number of voters or a budget constraint like in Che and Gale (1996). This is ruled out in our model, but as in Levine and Mattozzi (2019) we can approximate the effect by assuming that cost grows rapidly, and in particular becomes greater than the value of the prize, as the limiting effort level is approached. For these approximations our assumptions are satisfied so our results hold.

More generally, a model with a truncated effort level is equivalent to a model in which cost is discontinuous at the truncation point, jumping to a level greater than the value of the prize. Specifically, we now wish to consider the possibility that c_j instead of being continuous on the whole support, it is continuous on $[0, \bar{e}_j]$ where $\bar{e}_j > 0$, $c_j(\bar{e}_j) = \bar{c}_j < 1$, and for $e_j > \bar{e}_j$ we have $c_j(e_j) = c_{Max} > 1$.

Here it is crucial to emphasize that we did not use the continuity of the cost function in proving our results on advantage, so those results extend to this more general class of models. Furthermore, if the contest success function itself is continuous, we show in Theorem 17 that our robustness results continue to hold. This leaves the issue of robustness when both p and c are discontinuous, and here we can go no further. The following example adapted from Levine and Mattozzi (2019) shows that upper hemi-continuity of the equilibrium correspondence can fail in that case.

Example 3. Let the contest success function be that of the all-pay auction, and fix a cost function for both contestants that is linear with constant unit marginal cost up to a resource limit of $\bar{e}_j > 0$. We normalize the value of the prize to be 1, assume that $\bar{e}_j + \bar{e}_{-j} = 1/2$ and let $c_{Max} = 2$. This means that both contestants want to bid more than their resource limits.

Suppose first that $\bar{e}_1 > \bar{e}_{-1}$. In this case contestant 1 receives a utility of at least $3/4$ and contestant -1 gets a utility of 0. Moreover, it is well-known that in the unique equilibrium bidding is uniform in $(0, \bar{e}_{-1})$ and that -1 has an atom at zero and 1 has an atom at \bar{e}_{-1} . The implication of the non-trivial atom at \bar{e}_{-1} means, however, that the tie-breaking rule that each contestant has an equal chance of winning in case of a tie is not consistent with equilibrium. If that is the tie-breaking rule, then -1 should bid \bar{e}_{-1} , guaranteeing at least a 50% chance of winning, and so earning at least $1/2 - 1/4 > 0$ rather than zero as the equilibrium requires. In fact the tie-breaking rule at \bar{e}_{-1} must favor contestant 1 at least to the extent that contestant -1 cannot profit from that bid. In other words: when both p and c are discontinuous we must allow for endogenous tie-breaking rules.

Second, consider what happens as we pass through the point of symmetry. For $\bar{e}_1 > \bar{e}_{-1}$ contestant 1 earns at least $3/4$ and contestant -1 earns nothing; at $\bar{e}_1 = \bar{e}_{-1}$ both contestants earn $1/4$, while for $\bar{e}_1 < \bar{e}_{-1}$ contestant 1 earns 0 and contestant -1 earns at least $3/4$. In other words: both the individual and aggregate payoff are discontinuous as we pass through the point of symmetry.

Finally, suppose that we approximate the discontinuous cost functions by functions that are linear up to $\bar{e}_j - \epsilon$ then rise steeply to c_{Max} as \bar{e}_j is approached. Levine and Mattozzi (2019) show that in this case as long as $\bar{e}_1 \neq \bar{e}_{-1}$ payoffs are well-behaved in the limit. However, this is not the case when there is symmetry. If $\bar{e}_1 = \bar{e}_{-1}$ then with continuous cost there is complete rent dissipation: both players get zero. However, in the limit both contestants get $1/2 - 1/4$ so we have equilibria with complete rent dissipation converging to one where both contestants get a positive rent.

9. Examples in the Literature

A great many different combinations of conflict success functions and costs have been discussed in the literature. Before reviewing some of these examples, we start with a simple observation. Given the separability of the payoff function, the units of effort do not matter. While it might be natural from an economic point of view to identify effort with number of voters, hours devoted to the cause, or amount of money contributed, the model does not care about the units. Specifically, if we let $h(e)$ denote a continuous strictly increasing function with $h(0) = 0$, that is, a strictly increasing cost function, then the contest $p(h(e_j), h(e_{-j})), c_j(h(e_j))$ is *equivalent* to the contest $p(e_j, e_{-j}), c_j(e_j)$ in the sense that any equilibrium in one contest can be transformed to an equilibrium of the other contest with exactly the same probabilities of winning and costs. If an equilibrium strategy of the h contest is denoted by $F_{h_j}(e_j)$ we can map the

equilibrium strategies by $F_{h_j}(h(e_j)) = F_j(e_j)$ and $F_{h_j}(e_j) = F_j(h^{-1}(e_j))$. In particular, if for contestant -1 the cost $c_{-1}(e_{-1})$ is strictly increasing, we can take $h(e_{-j}) = c_{-1}^{-1}(e_{-j})$ in which case the cost function of -1 is linear and given by $c_{h^{-1}}(e_j) = e_j$. Notice the implication that the statement “cost is convex” has no real meaning in a contest: we can change the units of cost so as to make cost concave or convex and get an equivalent contest by suitably modifying the contest success function.

In a similar vein, not all models in the literature assume that the contest success function is symmetric. Many that do not, however, assume that one contestant has a bidding advantage. Let $h(e_1)$ be a strictly increasing continuous function with $h(e_1) \geq e_1$. This represents the idea that effort by 1 is “worth more” than effort by -1 : for example, in a political contest because 1 has a more appealing platform or more attractive candidate. Here the probability of winning for 1 is given by $p(h_1(e_1), e_{-1})$. However, there is a symmetric contest that is equivalent. Here we create a new contest with contest success given by $p(e_1, e_{-1})$ and redefine cost for 1 as $c_1(h^{-1}(e_1))$ for $e_1 \geq h(0)$ and 0 otherwise. Since it may be that there is a bidding advantage even at zero we need to allow the possibility that c_1 is flat up to $h(0)$.

Finally, in some settings the objective function is given as $p(e_j, e_{-j}) - c(e_j) - b(e_{-j})$ where $b(e_{-j})$ represents costs of $-j$ that are incurred by j or alternatively any collateral damage inflicted on j by the effort of $-j$. The point here is that subtracting a function that depends only on the other contestant’s choice changes the payoffs of the game but has no effect on equilibrium strategies. In particular, if we agree that it is still sensible to measure achievement of goals by $p(e_j, e_{-j}) - c(e_j)$, that is net of collateral damage about which j can do nothing, then $b(e_{-j})$ does not matter. It does matter, however, for assessment of the efficiency of different contest success functions: low collateral damage is obviously socially desirable. For example, in choosing between an election and a military conflict both designed to exactly mimic the contest success, the former is preferred because it avoids collateral damage.

In the literature contest success functions are derived from a variety of considerations. Some authors derive conflict resolution functions from axioms: see, for example, Jia, Skaperdas and Vaidya (2013). In the voting literature contest success functions are often derived from models of voter turnout where there are random shocks to voter preferences, see, for example, Levine and Mattozzi (2019) and references therein.

The analysis of equilibrium is also quite varied. Some authors restrict attention to pure strategy equilibria, see, for example, the survey of Corchón (2007). Some authors compute pure strategy equilibrium using first order conditions without checking second order conditions: their results only hold in part of the parameter space. Others impose specific parameter assumptions to assure the existence of pure strategy equilibrium, while yet others analyze mixed strategy equilibria. While contest success functions and the type of equilibrium analyzed are quite varied, assumptions about cost are less so: while not all authors assume linear cost it is certainly the most popular assumption.

The simplest and most studied example of a contest is the all-pay auction

in which the highest effort wins: if $e_j > e_{-j}$ then $p(e_j, e_{-j}) = 1$. Hillman and Riley (2006) gave a complete solution for linear cost while the case of non-linear cost as been covered by Siegel (2014) and Levine and Mattozzi (2019) among others. In the all-pay auction in which the highest effort wins the structure of equilibrium strategies and payoffs is well understood. There is a unique mixed strategy equilibrium. Payoffs are equivalent to those in a second price auction: the contestant with the least willingness to bid gets nothing and the contestant with the greatest willingness to bid gets the difference between the value of the prize and the cost of matching the willingness to bid of the opponent. In the linear cost case the equilibrium is easily described: the contestant 1 with the greatest willingness to bid chooses effort uniformly on $[0, w_{-1}]$ and the contestant with the least willingness to bid randomizes between providing zero effort and playing uniformly on $[0, w_{-1}]$ and the probability of zero effort is easily computed.

A second widely used contest success function is that of Tullock (1967). For a parameter $\beta \geq 0$ it is given by

$$p(e_j, e_{-j}) = \frac{e_j^\beta}{e_j^\beta + e_{-j}^\beta}.$$

This converges (pointwise) to the all-pay auction as $\beta \rightarrow \infty$. For $\beta \leq 1$ and weakly convex cost there is a unique pure strategy equilibrium, which is easy to compute in the linear cost case. The non-linear cost case has been studied as well: see, for example, Herrera, Morelli and Nunnari (2015). With linear cost for $\beta \geq 2$ there are only mixed strategy equilibria and these have been characterized in the symmetric case by Ewerhart (2015).

The Tullock contest success function is a special case of a contest success function that depends only on the *ratio* of efforts: $p(e_j, e_{-j}) = P(e_j/e_{-j})$. Notice that such a function is either constant or discontinuous at zero: if the probability of success is different for two different ratios then taking both levels of effort to zero holding fixed the ratio gives a different answer for the two. This implies, as we will indicate later, that there is no equilibrium in which both contestants provide zero effort with probability one. One example of a ratio contest success function different from the Tullock function is Feddersen and Sandroni (2006) who take $p(e_j, e_{-j}) = (1/2)(e_j/e_{-j})$ for $e_j < e_{-j}$. We note in passing that they assume also a linear effort advantage for one contestant and quadratic cost. They also assume collateral damage in that each contestant is assumed to care equally about the cost of the other contestant as about their own. As indicated above this matters only for efficiency computations, and other models of ethical voters (also known as rule utilitarian) such as Coate and Conlin (2004) have dropped that assumption.

An alternative to the Tullock function is the logit function introduced by Hirshleifer (1989)

$$p(e_j, e_{-j}) = \frac{\exp(\beta e_j)}{\exp(\beta e_j) + \exp(\beta e_{-j})}.$$

This also converges to the all-pay auction as $\beta \rightarrow \infty$. Hirshleifer (1989) originally solved the case with linear cost where β is small enough to allow a pure strategy equilibrium in which case the contestant with higher cost bids zero. While the Tullock function is a special case of a ratio model, the Hirshleifer function is a special case of a *difference* model

$$p(e_j, e_{-j}) = P(e_j - e_{-j}).$$

Difference models have been widely used, in part because they are well behaved with respect to a linear bidding advantage $h(e_1) = h_0 + e_1$. This is the case in Shachar and Nalebuff (1999) who take

$$p_j(e_j, e_{-j}) = H \left(\frac{1}{2} + \frac{\exp(e_j) - \exp(e_{-j})}{\exp(e_j) + \exp(e_{-j})} \right)$$

where H is a cdf on $[0, 1]$. In addition to a linear bidding advantage they allow also for an advantage in the H function. That type of advantage is not a special case of the symmetric contest success model: their model can only be reduced to a standard contest if the cdf H is symmetric around $1/2$. In a similar way the model of Coate and Conlin (2004) maps to a standard contest only if the parties are of equal expected size. By contrast Herrera, Levine and Martinelli (2008) allow only bidding advantage so their model is equivalent to a standard contest for all parameter values.

The papers using difference models discussed so far restrict attention to pure strategy equilibria. In the case of a *quasi-linear* function $P(e_j - e_{-j})$ which is linear when it is not 0 or 1 a substantial amount is known about mixed strategy equilibrium. Che and Gale (2000) give a relatively complete analysis when cost is linear and explicitly compute the equilibria, which all have finite support. Ashworth and Bueno De Mesquita (2009) extend that analysis to the case where one contestant has linear cost and the other has a linear effort advantage. The fact that equilibria have finite support is not that surprising in light of our subsequent results in Theorem 9. The quasi-linear function is not differentiable. If we take the opposite case of a conflict resolution function (and cost functions) that are real analytic, we show that the support for both players is always finite. In particular, this is the case for difference models such as the logit of Hirshleifer (1989), or for example, if we take P to be a normal cdf.

Most of the models in the literature are either of the ratio or difference form. Despite some controversy over which is more appropriate the two are not so different. For starters, in both Hirshleifer (1989) and Tullock (1967) if β is large both are similar to an all-pay auction. Since the equilibrium of the all-pay auction is unique our Theorem 7 shows that the equilibrium strategies for both models must be similar in the sense that the difference between any two must converge weakly to zero. The models are more similar than that, however. If in a ratio model we allow a small effort for free, so that $c_j(e_j) = 0$ for $0 \leq e_j \leq b$ for some $b > 0$ then since the strategy of providing less than b units of effort is weakly dominated (strictly if p is strictly increasing) we may as well assume that

the contestants make effort only $e_j \geq b$ and we can then change the units so that b in the original units corresponds to 0 in the new units, that is, $h(e_j) = e_j + b$ (see for example Amegashi (2006)), in which case we have for the ratio model

$$P\left(\frac{e_j + b}{e_{-j} + b}\right).$$

Having eliminated by a small perturbation the discontinuity at 0 we can now change units again with $h(e_j) = b(\exp(e_j) - 1)$ to convert the ratio model into

$$P\left(\frac{\exp(e_j)}{\exp(e_{-j})}\right),$$

an equivalent difference model.

We conclude with an observation about Hirshleifer (1989)'s argument that one contestant should be expected to provide zero effort in any equilibrium. He points out that it is likely to be the case in practice that effort should make the greatest difference when the contest is close. Call a contest success function *convex-concave* if $p(e_j, e_{-j})$ as a function of e_j is strictly convex for $e_j < e_{-j}$ and strictly concave for $e_j > e_{-j}$. This captures the idea that the closer the contest the more effort matters. It fails in the Tullock model since, as Hirshleifer points out, $p(e_j, e_{-j})$ as a function of e_j has an inflection point at $e_j < e_{-j}$. We can generalize Hirshleifer's result with linear costs to more general convex-concave function and mixed strategy equilibria by adapting his argument.

Theorem 11. *Suppose the contest success function is real analytic and convex-concave and that costs are linear. Then one contestant provides zero effort with strictly positive probability.*

Proof. Theorem 9 shows that the support of the equilibrium strategies are finite. Suppose that the lowest point \underline{e}_j of support in the equilibrium F_j is no greater than that in F_{-j} . Then $p(e_j, F_{-j})$ must be twice continuously differentiable and convex over $0 \leq e_j \leq \underline{e}_j$. This implies the same is true for the payoff function $p(e_j, F_{-j}) - C_j e_j$. Unless $\underline{e}_j = 0$ this is inconsistent with \underline{e}_j being optimal for j . \square

This result is somewhat less interesting in the mixed strategy case since the contestant that provides zero effort may also with positive probability provide the highest effort.

10. Conclusion

The goal of this paper has been to establish general results about contests. We characterize cost functions for which there are peaceful and contested equilibria. We then prove three main results. First, there cannot be a pre-emptive equilibrium in which the higher cost contestant has greater success. Second, a contestant with a sufficiently great cost advantage always has greater success.

Third, if the cost advantage is a homogeneous one, then the lower cost contestant always has greater success. Finally, we study the robustness of equilibrium. We prove a basic upper hemi-continuity result and examine approximation by real analytic functions. This enables us to show that properties involving strict inequality are robust and that large classes of examples have equilibria with finite support.

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11. Appendix: Upper Hemi-Continuity

Mathematical Preliminaries

Suppose that X is a compact rectangle in \mathbb{R}^M , that $f_n(x), f_0(x)$ are uniformly bounded non-decreasing real valued functions on X such that $f_n(x) \rightarrow f_0(x)$. Denote by D the set of discontinuities of $f_0(x)$ and by \bar{D} the closure of D .

Theorem 12. *Suppose that $D_o \supset \bar{D}$ is an open subset of X . Then f_n converges uniformly to f on $X \setminus D_o$.*

Proof. If $X \setminus D_o$ is empty this is true trivially. Otherwise as $X \setminus D_o$ is compact if the theorem fails there is a sequence $x_n \in X \setminus D_o$ with $x_n \rightarrow x \in X \setminus D_o$ and $f_n(x_n) \rightarrow z \neq f_0(x)$. There are two cases as $z < f_0(x)$ and $z > f_0(x)$. Denote the bottom corner of X as y_0 and the top corner as y_1 . Notice that since D_o is open and contains the closure of D , then x has an open neighborhood in which f_0 is continuous.

If $z < f_0(x)$ and $x \neq y_0$ since f_0 is continuous near x there is a $y < x$ with $f_0(y) > z$ and an N such that for $n > N$ we have $x_n > y$. Since f_n is non-decreasing $f_n(x_n) \geq f_n(y)$. Hence $z \geq f_0(y)$ a contradiction. If $x = y_0$ then $f_n(y_0) \rightarrow f_0(y_0)$ while $f_n(x_n) \geq f_n(y_0)$. Taking limits on both sides we get $z \geq f_0(y_0)$ a contradiction

If $z > f_0(x)$ and $x \neq y_1$ we have $y > x$ such that $f_0(y) < z$ and an N such that for $n > N$ we have $x_n < y$. This gives $f_n(x_n) \leq f_n(y)$ implying $z \leq f_0(y)$ a contradiction. If $x = y_1$ we have $f_n(x_1) \rightarrow f_0(x_1)$ and $f_n(x_n) \leq f_n(x_1)$ and taking limits on both sides we get $z \leq f_0(x_1)$ a contradiction. \square

We say that an open set D_o *encompasses* f_0 if there is a closed set $D_1 \subset D_o$ such that the interior of D_1 contains \bar{D} . Let \bar{D}_o denote the closure of D_o .

Theorem 13. *Suppose that the probability measures μ_n converge weakly to μ_0 . If there is a sequence of sets D_a^m, D_g^m with $D_a^m \cup D_g^m$ encompassing f_0 such that $\limsup_m \limsup_n \sup_{x \in \bar{D}_a^m} |f_n(x) - f_0(x)| = 0$ and $\limsup_m \limsup_n \mu_n(\bar{D}_g^m) = 0$ then $\lim \int f_n d\mu_n = \int f_0 d\mu_0$.*

Proof. By Urysohn's Lemma there are continuous functions $0 \leq g^m(x) \leq 1$ equal to 1 for $x \in X \setminus D_0^m$ and equal to zero for $x \in D_1^m$. Setting $D_o^m = D_g^m \cup D_a^m$

$$\begin{aligned} \left| \int f_n d\mu_n - \int f_0 d\mu_0 \right| &\leq \left| \int g^m f_n d\mu_n - \int g^m f_0 d\mu_0 \right| + \\ &+ \left| \int (1 - g^m) f_n d\mu_n - \int (1 - g^m) f_0 d\mu_0 \right| \leq \\ &\leq \left| \int g^m f_n d\mu_n - \int g^m f_0 d\mu_0 \right| + \left| \int_{\bar{D}_0^m} f_n d\mu_n - \int f_0 d\mu_0 \right|. \end{aligned}$$

If ϕ_n, ϕ_0 are real numbers and m_n, m_0 are non-negative real numbers we have the inequality $|\phi_n m_n - \phi_0 m_0| \leq |\phi_n - \phi_0|(m_n + m_0)$ so

$$\left| \int f_n d\mu_n - \int f_0 d\mu_0 \right| \leq \left| \int g^m f_n d\mu_n - \int g^m f_0 d\mu_0 \right| + \int_{\overline{D}_o^m} |f_n - f_0| d(\mu_n + \mu_0).$$

First we show that $\int_{\overline{D}_o^m} |f_n - f_0| d(\mu_n + \mu_0) \rightarrow 0$. Let $\bar{f} = \sup |f_k(x)|$. we have

$$\begin{aligned} \int_{\overline{D}_o^m} |f_n - f_0| d(\mu_n + \mu_0) &\leq \int_{\overline{D}_a^m} |f_n - f_0| d(\mu_n + \mu_0) + \int_{\overline{D}_g^m} |f_n - f_0| d(\mu_n + \mu_0) \\ &\leq \sup_{x \in \overline{D}_a^m} |f_n(x) - f_0(x)| + \bar{f} \left(\mu_n(\overline{D}_g^m) + \mu_0(\overline{D}_g^m) \right). \end{aligned}$$

The first term converges to 0 by hypothesis. For the second, as \overline{D}_g^m is closed and μ_n converges weakly to μ_0 we have $\mu_0(\overline{D}_g^m) \leq \limsup \mu_n(\overline{D}_g^m)$ so

$$\limsup_n \bar{f} \left(\mu_n(\overline{D}_g^m) + \mu_0(\overline{D}_g^m) \right) \leq 2\bar{f} \limsup_n \mu_n(\overline{D}_g^m)$$

giving the first result. Second, write

$$\left| \int g^m f_n d\mu_n - \int g^m f_0 d\mu_0 \right| \leq \left| \int g^m |f_n - f_0| d\mu_n \right| + \left| \int g^m f_0 d\mu_0 - \int g^m f_0 d\mu_n \right|.$$

Since $g^m f_0$ is continuous by construction we have $\lim_n \left| \int g^m f_0 d\mu_0 - \int g^m f_0 d\mu_n \right| = 0$ by weak convergence of μ_n to μ_0 .

Finally, we show that $\lim_n \int g^m |f_n - f_0| d\mu_n = 0$. Denote by $D_1^{m_o}$ the interior of D_1^m and $X_1^m = X \setminus D_1^{m_o}$. By Theorem 12 $|f_n(x) - f_0(x)| \leq \epsilon_n^m$ for $x \in X_1^m$ where $\lim_n \epsilon_n^m = 0$. As $g^m(x) = 0$ for $x \in D_1^m \supset D_1^{m_o}$ we have $g^m |f_n - f_0| \leq \epsilon_n^m$ so that $\int g^m |f_n - f_0| d\mu_n \leq \epsilon_n$. \square

Recall that \overline{D} denote the closure of D .

Theorem 14. *Suppose that X is a compact rectangle in \mathbb{R}^M , that $f_n(x), f_0(x)$ are uniformly bounded non-decreasing real valued functions on X , that $f_n(x) \rightarrow f_0(x)$ and that the probability measures μ_n converge weakly to μ_0 . If $\mu_0(\overline{D}) = 0$ then $\lim \int f_n d\mu_n = \int f_0 d\mu_0$.*

Proof. Take the sets $D_o^m = D_g^m$ to be the open $\epsilon_m \rightarrow 0$ neighborhoods of \overline{D} and take $D_a^o = \emptyset$. We may take D_1^m sets to be the closed $\epsilon/2$ neighborhoods of \overline{D} : this clearly contains \overline{D} in its interior and is contained in D_o^m . Take D_2^m to be the open $2\epsilon_m$ neighborhoods of D : as these contain \overline{D}_o^m it suffices to show that $\limsup_m \limsup_n \mu_n(D_2^m) = 0$. Since D_2^m is open and μ_n converges weakly to μ_0 we have $\limsup_n \mu_n(D_2^m) \leq \mu_0(D_2^m)$, so we need only prove $\limsup_m \mu_0(D_2^m) = 0$. Since $\cap_m D_2^m = \overline{D}$ we have $\lim_m \mu_0(D_2^m) = \mu_0(\overline{D}) = 0$. \square

Upper Hemi-Continuity of the Equilibrium Correspondence

We now consider a *convergence scenario*. Here $p_n(e_1, e_{-1}) \rightarrow p_0(e_1, e_{-1})$, $c_{jn}(e_j) \rightarrow c_{j0}(e_j)$ is a sequence of contests on W . We take F_{1n}, F_{-1n} to be

equilibria for n converging weakly to F_{10}, F_{-10} with μ_{jk} the corresponding measures. We say that the convergence scenario is *upper hemi-continuous* if $p_n(F_{jn}, F_{-jn}) \rightarrow p_0(F_{j0}, F_{-j0})$, $c_{jn}(F_{jn}) \rightarrow c_{j0}(F_{j0})$ for both j and F_{10}, F_{-10} is an equilibrium for $p_0(e_1, e_{-1}), c_{j0}(e_j)$.

Lemma 4. *If $p_n(F_{jn}, F_{-jn}) \rightarrow p_0(F_{j0}, F_{-j0})$ for both j then the convergence scenario is upper hemi-continuous.*

Proof. By Theorem 14 $c_{jn}(F_{jn}) \rightarrow c_{j0}(F_{j0})$ on the relevant domain $0 \leq e_j \leq W$. This shows that $u_{jn}(F_{jn}, F_{-jn}) \rightarrow u_{j0}(F_{j0}, F_{-j0})$. Next consider j deviating to $e_j \in [0, W]$. Suppose first that e_j is an atom of F_{-j0} . Then this is not a best response. Suppose second that e_j is not an atom of F_{-j0} . Hence the function of e given by $p_0(e_j, e)$ has measure zero with respect to F_{-j0} . It follows from Theorem 14 that $p_n(e_j, F_{-1n}) \rightarrow p_0(e_j, F_{-j0})$, so also $u_{jn}(e_j, F_{-jn}) \rightarrow u_{j0}(e_j, F_{-j0})$. If e_j was a profitable deviation, that is, $u_{j0}(e_j, F_{-j0}) > u_{j0}(F_{j0}, F_{-j0})$, it follows by the standard argument that for sufficiently large n we would have $u_{jn}(e_j, F_{-jn}) > u_{jn}(F_{jn}, F_{-jn})$ contradicting the optimality of F_{jn} . \square

In what follows all sequences are of strictly positive numbers.

Lemma 5. *If $\gamma^m \rightarrow 0$ then there are sequences $G^n, H^m \rightarrow 0$ such that on $[0, W + 2 \max \gamma^m]$ we have $\max_{e \in [0, W]} c_{jn}(e + 2\gamma^m) - c_{jn}(e) \leq G^n + H^m$.*

Proof. By Lemma 12 we have c_{jn} converging uniformly to c_{j0} so that

$$\max_{e \in [0, W]} c_{jn}(e + 2\gamma^m) - c_{jn}(e) \leq \max_{e \in [0, W]} c_{j0}(e + 2\gamma^m) - c_{j0}(e) + G_{jn}$$

Since c_{j0} is uniformly continuous on compact intervals $\max_{e \in [0, W]} c_{j0}(e + 2\gamma^m) - c_{j0}(e) \leq H_{jm}$. Then take $G^n = \max G_{jn}, H^m = \max H_{jm}$. \square

Lemma 6. *Fix sequences $\gamma^m, \theta^m \rightarrow 0$. Then there exists a sequence $u^n \rightarrow 0$ and $\gamma^m \geq \omega^m$ such that for $0 \leq e_{-j} - e \leq \omega^m$:*

(i) *If $p(e + \gamma^m) - 1/2 < \theta^m$ then $\sup_{0 \leq e_k - e \leq \omega^m} |p_n(e_j, e_{-j}) - p_0(e_j, e_{-j})| \leq 2\theta^m + u^n$.*

(ii) *If $p(e + \gamma^m) - 1/2 \geq \theta^m$ then $p_n(e + \gamma^m + \omega^m, e_{-j}) - 1/2 \geq \theta^m/2 - u^n$.*

Proof. We may apply Theorem 13 to the functions $p_n(e_j, -x_{-j}), p_0(e_j, -x_{-j})$ on the rectangle $[0, W] \times [-W, 0]$ with $D_o = \{(e_j, x_{-j}) \mid |e_j + x_{-j}| < \gamma^m\}$ to conclude that $p_n(e_j, -x_{-j})$ converges uniformly to $p_0(e_j, -x_{-j})$ there. Hence there exists a constant u^m such that for $e_j - e_{-j} \geq \gamma^m$ we have $|p_n(e_j, e_{j-1}) - p_0(e_j, e_{j-1})| \leq u^n$.

Fix e . For (i) Take $\omega^m = \gamma^m$. Take $0 \leq e_k - e \leq \omega^m$. Observe that

$$p_0(e_j, e_{-j}) \leq p_0(e + \omega^m, e) < 1/2 + \theta^m.$$

Since $e + \omega^m - e \geq \gamma^m$ we also have $|p_n(e + \omega^m, e) - p_0(e + \omega^m, e)| \leq u^n$ this implies

$$p_n(e_j, e_{-j}) \leq 1/2 + \theta^m + u^n.$$

Reversing the role of j and $-j$ we see that

$$|p_0(e_j, e_{-j}) - 1/2| < \theta^m, |p_n(e_j, e_{-j}) - 1/2| < \theta^m + u^n.$$

Hence $|p_n(e_j, e_{-j}) - p_0(e_j, e_{-j})| < 2\theta^m + u^n$.

For (ii), observe that $p_0(e_j, e_{-j})$ is uniformly continuous on $e_j - e_{-j} \geq \gamma^m$. Hence we may find a $\omega^m > 0$ which without loss of generality we may take to be smaller than γ^m such that for $|e_{-j} - e| \leq \omega^m$ we have $|p_0(e_j, e_{-j}) - p_0(e_j, e)| < \theta^m/2$. Since $p_n(e + \gamma^m + \omega^m, e_{-j})$ is non-increasing in e_{-j} we put this all together:

$$\begin{aligned} p_n(e + \gamma^m + \omega^m, e_{-j}) &\geq p_n(e + \gamma^m + \omega^m, e + \omega^m) \geq p_0(e + \gamma^m + \omega^m, e + \omega^m) - u^n \\ &\geq p_0(e + \gamma^m + \omega^m, e) - \theta^m/2 - u^n \geq p(e + \gamma^m) - \theta^m/2 - u^n \geq 1/2 + \theta^m/2 - u^n. \end{aligned}$$

□

Lemma 7. *For any $\gamma^m \rightarrow 0$ there are sequences $G^n, H^m \rightarrow 0$ such that for any θ^m and $\omega^m \leq \gamma^m$ and any e with $p_n(e + \gamma^m + \omega^m, e_{-j}) - 1/2 \geq \theta^m/2 - u^n > 0$ for all $0 \leq e_{-j} - e \leq \omega^m$ we have*

$$\min_j \mu_{jn}([e, e + \omega^m]) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}$$

Proof. Given $\gamma^m \rightarrow 0$ choose the sequences G^n, H^m by Lemma 5.

Define $m_j \equiv \mu_{jn}([e, e + \omega^m])$. If for one j we have $m_j = 0$ then certainly the inequality holds. Otherwise, consider that if each j plays μ_{jn}/m_j in $[e, e + \omega^m]$ then one of them must have probability no greater than $1/2$ of winning. Say this is j . Consider the strategy for j of switching from μ_{jn} to $\hat{\mu}_{jn}$ by not providing effort in $[e, e + \omega^m]$ and instead providing effort with probability m_j at $e + \gamma^m + \omega^m$. This results in a utility gain of at least

$$\begin{aligned} &m_{-j}(\theta^m/2 - u^n) - (c_{jn}(e + \gamma^m + \omega^m) - c_{jn}(e)) \\ &\geq m_{-j}(\theta^m/2 - u^n) - (c_{jn}(e + 2\gamma^m) - c_{jn}(e)) \geq m_{-j}(\theta^m/2 - u^n) - (G^n + H^m). \end{aligned}$$

As the utility gain cannot be positive, this implies $0 \geq m_{-j}(\theta^m/2 - u^n) - (G^n + H^m)$ giving the desired inequality. □

Theorem 15. *Convergence scenarios are upper hemi-continuous.*

Proof. By Lemma 4 it suffices to show $p_n(F_{jn}, F_{-jn}) \rightarrow p_0(F_{j0}, F_{-j0})$.

Observe that $p_n(e_j, e_{-j}), p_0(e_j, e_{-j})$ are non-decreasing in the first argument and non-increasing in the second so that the functions on the rectangle $[0, W] \times [-W, 0]$, given by $f_k(x) \equiv p_k(x_j, -x_{-j})$, are uniformly bounded. Define $\mu_n = \mu_{1n} \times \mu_{-1n}$ and $\mu_0 = \mu_{10} \times \mu_{-10}$. From Fubini's Theorem μ_n converges weakly to μ_0 . so Theorem 13 applies if we can show how to construct the sets D_a^m, D_g^m .

Fix a sequence $\gamma^m \rightarrow 0$. Choose sequences G^n, H^m by Lemma 7 and choose $\theta^m \rightarrow 0$ so that $H^m/\theta^m \rightarrow 0$. Then choose $u^n \rightarrow 0$ and $\omega^m \leq \gamma^m$ by Lemma 6.

We cover the diagonal with open squares of width ω^m . Specifically, for $\ell = 1, 2, \dots, L$ we take the lower corners κ_ℓ of these squares to be $0, 2\omega^m/3, 4\omega^m/3, \dots$ until the final square overlaps the top corner at (W, W) . There are two types of squares: a -squares where $p(\kappa_\ell + \gamma^m) - 1/2 < \gamma^m$ and g -squares where $p(\kappa_\ell + \gamma^m) - 1/2 \geq \gamma^m$.

We take D_a^m to be the union of the a -squares and D_g^m to be the union of the g -squares. Then for each square ℓ we may take a closed square with the same corner but $3/4$ ths the width and define D_1 to be the union of these squares. Then $D_o^m = D_a^m \cup D_b^m \supset D_1 \supset \bar{D}$ so that indeed D_o^m encompasses p_0 .

Since D_a^m is the union of a -squares, by Lemma 6 (i) we have $\sup_{x \in \bar{D}_a^m} |f_n(x) - f_0(x)| \leq 2\theta^m + u^n$, so indeed $\limsup_m \limsup_n \sup_{x \in \bar{D}_a^m} |f_n(x) - f_0(x)| = 0$ as required by Theorem 13.

For a g -square ℓ we have $0 \leq e_{-j} - e \leq \omega^m$ so by Lemma 6 $p_n(e + \gamma^m + \omega^m, e_{-j}) - 1/2 \geq \theta^m/2 - u^n$. Then by Lemma 7

$$\min_j \mu_{jn}([\kappa_\ell, \kappa_\ell + \omega^m]) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}.$$

We now add up over the g -squares four times, once for the odd numbered ones and once for the even numbered ones. This assures that each sum is over disjoint squares. In each case we first add those for which $j = 1$ has the lowest value of $\mu_{jn}([\kappa_\ell, \kappa_\ell + \omega^m])$ and once for $j = -1$. In each set of indices Λ we get a sum

$$\begin{aligned} \sum_{\ell \in \Lambda} \mu_{jn}([\kappa_\ell, \kappa_\ell + \omega^m]) \mu_{-jn}([\kappa_\ell, \kappa_\ell + \omega^m]) &\leq \\ \frac{G_n + H_m}{\theta^m/2 - u^n} \sum_{\ell \in \Lambda} \mu_{-jn}([\kappa_\ell, \kappa_\ell + \omega^m]) &\leq \frac{G_n + H_m}{\theta^m/2 - u^n}. \end{aligned}$$

This gives a bound

$$\mu_n(\bar{D}_g^m) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}.$$

We then have

$$\limsup_n \mu_n(\bar{D}_g^m) \leq \frac{H_m}{\theta^m/2}$$

and since we constructed the sequences so that $H^m/\theta^m \rightarrow 0$ the result now follows from Theorem 13. \square

12. Appendix: Smoothing Conflict Resolution Functions

Theorem 16. *If p, c_j is a contest on W then there is a sequence of well-behaved contests p_n, c_{jn} on W with $p_n(e_j, e_{-j}) \rightarrow p(e_j, e_{-j}), c_{jn}(e_j) \rightarrow c_j(e_j)$ for every $(e_1, e_{-1}) \in [0, W] \times [0, W]$.*

To prove this theorem we first state and prove

Lemma 8. *Suppose that $p_n(e_j, e_{-j}) \rightarrow p_0(e_j, e_{-j})$ and $p_{mn}(e_j, e_{-j}) \rightarrow_m p_n(e_j, e_{-j})$. Then there is $M(n)$ such that $p_{M(n)n}(e_j, e_{-j}) \rightarrow p_0(e_j, e_{-j})$.*

Proof. Define $d(p, q) = \inf\{\gamma \mid \sup_{|e_j - e_{-j}| \geq \gamma} |p(e_j, e_{-j}) - q(e_j, e_{-j})| \leq \gamma\}$. Then $d(p, q) = 0$ if and only if $p = q$, $d(p, q) = d(q, p)$ and $d(p, q) + d(q, r) \leq 2 \max\{d(p, q), d(q, r)\}$. Moreover, $d(p_n, p_0) \rightarrow 0$ if and only if $p_n(e_j, e_{-j}) \rightarrow p_0(e_j, e_{-j})$. Let $\epsilon_n \rightarrow 0$ and take $M(n)$ such that for $m \geq M(n)$ we have $d(p_{mn}, p_n) < \epsilon_n$. Then $d(p_{M(n)n}, p_0) \leq 2 \max\{\epsilon_n, d(p_n, p_0)\} \rightarrow 0$. \square

We now prove Theorem 16.

Proof. By Lemma 8 we can do the perturbations sequentially.

Step 1: Perturb p to get it strictly increasing with strictly positive infimum: take $p_n(e_j, e_{-j}) = (1 - \lambda_n)p(e_j, e_{-j}) + \lambda_n\Phi(e_j - e_{-j})$ where Φ is the standard normal cdf.

Step 2: Given p strictly increasing and positive perturb it to get it strictly increasing, positive and C^2 . Let $g_n(x_j|e_j) = (1/W)h_n(x_j/W|e_j)$ where $h_n(\bullet|e_j)$ is the Dirichlet distribution with parameter vector

$$8n^3 \left[\left(1 - \frac{1/n}{2\sqrt{2}}\right)(e_j/W) + \frac{1/n}{2\sqrt{2}} \frac{1}{2} \right], 8n^3 \left[\left(1 - \frac{1/n}{2\sqrt{2}}\right)(1 - e_j/W) + \frac{1/n}{2\sqrt{2}} \frac{1}{2} \right].$$

This is C^∞ in b_j and $g_n(0|e_j) = g_n(W|e_j) = 0$ and taking $p_n(b_j, b_{-j}) \equiv \int_0^\infty p(x_j, x_{-j})g_n(x_j|b_j)g_n(x_{-j}|b_{-j})dx_j dx_{-j}$ this is certainly strictly positive and C^2 . To see that it is strictly increasing observe that increasing b_j increases $g_n(x_j|e_j)$ in first order stochastic dominance. Finally, it is shown in the Web Appendix of Dutta, Levine and Modica (2018) that $\Pr(|x_j - e_j| > 1/n) \leq 1/n$ so that we have pointwise convergence at every continuity point of p . Pointwise convergence on the diagonal is by definition.

Step 3: Given p strictly increasing, positive and C^2 perturb it to get it strictly increasing, positive on $[0, W] \times [0, W]$ and real analytic in an open neighborhood. By Whitney (1934) Theorem 1 we can extend p to be C^1 on all of R^2 . Take an open neighborhood \mathcal{W} of $[0, W] \times [0, W]$ so that p is strictly positive there. By Whitney (1934) Lemma 5 for each $\epsilon > 0$ we can find a real analytic function $q(b_j, b_{-j})$ with $|q - p| < \epsilon$ and $|Dq - Dp| < \epsilon$ on the closure of \mathcal{W} . Then define $Q(b_j, b_{-j}) = q(b_j, b_{-j}) / (q(b_j, b_{-j}) + q(b_{-j}, b_j))$.

Remark: The case of c_j is similar but easier. In the final step the real analytic function $q_j(b_j)$ is not necessarily zero at zero so we define $Q_j(b_j) = q_j(b_j) - q_j(0)$. \square

13. Appendix: Resource Limits

A *resource constrained contest* on W is a contest success function $p(e_j, e_{-j})$ together with a pair of cost functions $c_j(e_j)$ that satisfy the definition of being a contest except that p is required to be continuous and we allow the possibility that c_j instead of being continuous on the entire support is continuous on $[0, \bar{e}_j]$

where $\bar{e}_j > 0$, $c_j(\bar{e}_j) = \bar{c}_j < 1$, and for $e_j > \bar{e}_j$ we have $c_j(e_j) = c_{Max} > 1$. Our goal is to prove:

Theorem 17. *Suppose $p_n(e_1, e_{-1}) \rightarrow p_0(e_1, e_{-1})$, $c_{jn}(e_j) \rightarrow c_{j0}(e_j)$ for $e_j \neq \bar{e}_{j0}$ are a sequence of resource constrained contests in W , that F_{1n}, F_{-1n} are equilibria for n converging weakly to F_{10}, F_{-10} . Then $p_n(F_{jn}, F_{-jn}) \rightarrow p_0(F_{j0}, F_{-j0})$, $c_{jn}(F_{jn}) \rightarrow c_{j0}(F_{j0})$ for both j and F_{10}, F_{-10} is an equilibrium for $p_0(e_1, e_{-1})$, $c_{j0}(e_j)$.*

Proof. If c_{j0} is continuous then $c_{jn}(e_j) \rightarrow c_{j0}(e_j)$ for all e_j there is nothing new to be proven. We take then the discontinuous case. There are two new things that must be shown. First, we must show that if a deviation to \bar{e}_{j0} against F_{j0} is profitable then, because we do not have pointwise convergence at \bar{e}_{j0} , there is another deviation that is also profitable. Second, we must show that $c_{jn}(F_{jn}) \rightarrow c_{j0}(F_{j0})$.

The first is simple: if we take a sequence $e_{jm} \rightarrow \bar{e}_{j0}$ strictly from below, the continuity of p_0, c_{j0} imply that $u_{j0}(e_{jm}, F_{-j}) \rightarrow u_{j0}(\bar{e}_{j0}, F_{-j})$ so that for large enough m the deviation $e_{jm} \neq \bar{e}_{j0}$ is also profitable.

To prove the second we first choose $0 < \epsilon < (c_{Max} - 1)/2$. We observe that for each n (including $n = 0$) the fact that c_{jn} is weakly decreasing and left continuous means that $\{e_j | c_{jn}(e_j) \leq \bar{c}_{j0} + \epsilon\} = [0, e_{jn}(\epsilon)]$ and $\{e_j | c_{jn}(e_j) > \bar{c}_{j0} + \epsilon\} = (e_{jn}(\epsilon), W]$ where it is apparent that $e_{j0}(\epsilon) = \bar{e}_{j0}$. Moreover, we can show that $\lim_n e_{jn}(\epsilon) = \bar{e}_{j0}$. To see that for any $\gamma > \bar{e}_{j0}$ we have $\lim_n c_{jn}(\gamma) = c_{Max}$ implying $\limsup e_{jn}(\epsilon) \leq \gamma$. For any $\gamma < \bar{e}_{j0}$ we have $\lim_n c_{jn}(\gamma) \leq c_{j0}(\gamma) \leq \bar{c}_{j0}$ implying $\liminf_n e_{jn}(\epsilon) \geq \gamma$.

Second, since p_0 is continuous, pointwise convergence of p_n to p_0 implies uniform convergence and since W is compact, p_0 is uniformly continuous. It follows that $\Delta(\epsilon) = \inf\{0 \leq e_j^1 - e_j^2 | p_n(e_j^2, e_{-j}) - p_n(e_j^1, e_{-j}) \leq \epsilon\}$ is positive.

Third, we show that for sufficiently large n we have

$$\mu_{jn}((e_{jn}(\epsilon), \bar{e}_{j0} + \Delta(\epsilon/2)/2]) = 0.$$

Suppose that $e_j \in (e_{jn}(\epsilon), \bar{e}_{j0} + \Delta(\epsilon/2)/2]$. Then $c_{jn}(e_j) \geq \bar{c}_{j0} + \epsilon$ while $c_{jn}(\bar{e}_{j0} - \Delta(\epsilon/2)/2) \leq c_{j0}(\bar{e}_{j0} - \Delta(\epsilon/2)/2) + \eta_n$ where $\eta_n \rightarrow 0$. Since $e_j - (\bar{e}_{j0} - \Delta(\epsilon/2)/2) \leq \Delta(\epsilon/2)$ it follows that $p_n(e_j, F_{-j}) - p_n(\bar{e}_{j0} - \Delta(\epsilon/2)/2, F_{-j}) \leq \epsilon/2$, while $c_{jn}(e_j) - c_{jn}(\bar{e}_{j0} - \Delta(\epsilon/2)/2) \geq \epsilon - \eta_n$. Hence for $\eta_n < \epsilon/2$ it is not optimal to play e_j .

Fourth, we show that for sufficiently large n we have $\mu_{jn}((e_{jn}(\epsilon), W]) = 0$. To do so we need only show that for sufficiently large n we have $\mu_{jn}((\bar{e}_{j0} + \Delta(\epsilon/2)/2, W]) = 0$. Since $c_{jn}(\bar{e}_{j0} + \Delta(\epsilon/2)/2) \rightarrow c_{Max}$ for all sufficiently large n we have $c_{jn}(\bar{e}_{j0} + \Delta(\epsilon/2)/2) > 1$ and since c_{jn} is non-decreasing $c_{jn}(e_j) > 1$ for all $e_j \geq \bar{e}_{j0} + \Delta(\epsilon/2)/2$. Of course it cannot be optimal to play such an e_j .

Fifth we show that $\mu_{j0}((\bar{e}_{j0}, W]) = 0$. This follows from the fact that it is the countable union of the sets

$$(\bar{e}_{j0} + |e_{jn}(\epsilon) - \bar{e}_{j0}|, W] \subset (e_{jn}(\epsilon), W].$$

Sixth, we construct approximating functions \tilde{c}_{jn} . Since c_{j0} is continuous on $[0, \bar{e}_{j0}]$ we may choose $\gamma < \bar{e}_{j0}$ so that $c_{j0}(\bar{e}_{j0}) - c_{j0}(\gamma) < \epsilon$. Then for $e_j \leq \gamma$ we take $\tilde{c}_{jn}(e_j) = c_{jn}(e_j)$ and for $e_j > \gamma$ we take $\tilde{c}_{jn}(e_j) = c_{jn}(\gamma)$. Certainly then \tilde{c}_{jn} is non-decreasing and converges pointwise to the non-decreasing function \tilde{c}_{j0} . It follows that the convergence is uniform, hence $\tilde{c}_{jn}(F_{jn}) \rightarrow \tilde{c}_{0n}(F_{j0})$.

Seventh, we bound

$$\begin{aligned}
& |\tilde{c}_{jn}(F_{jn}) - c_{jn}(F_{jn})| \\
& \leq \int_{[0, \gamma]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})| dF_{jn} + \int_{(\gamma, e_{jn}(\epsilon)]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})| dF_{jn} \\
& \quad + \left| \int_{(e_{jn}(\epsilon), W]} (\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})) dF_{jn} \right| \\
& = \int_{(\gamma, e_{jn}(\epsilon)]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})| dF_{jn} \\
& \leq \sup_{(\gamma, e_{jn}(\epsilon)]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})| \\
& \quad = c_{jn}(e_{jn}(\epsilon)) - c_{jn}(\gamma) \\
& \leq |c_{jn}(e_{jn}(\epsilon)) - c_{j0}(\bar{e}_{j0})| + |c_{j0}(\bar{e}_{j0}) - c_{j0}(\gamma)| + |c_{j0}(\gamma) - c_{jn}(\gamma)| \\
& \leq 2\epsilon + \eta_n
\end{aligned}$$

where $\eta_n \rightarrow 0$.

Finally, we put this together to see that for all $0 < \epsilon < 1/2$ and sufficiently large n we have

$$|c_{jn}(F_{jn}) - c_{j0}(F_{j0})| \leq |\tilde{c}_{jn}(F_{jn}) - \tilde{c}_{j0}(F_{j0})| + 4\epsilon + 2\eta_n.$$

It follows that $\limsup |c_{jn}(F_{jn}) - c_{j0}(F_{j0})| \leq 4\epsilon$. This proves the result. \square

14. Appendix: Finite Support

Theorem 18. *Suppose that $c_1(b_j) = 0$ for $0 \leq b_1 \leq \underline{w}_1$ and if $w > 0$ we require that $p(b_j, b_{-j})$ is strictly increasing (so in particular in any equilibrium $\mu_1([0, \underline{w}_1]) = 0$). Suppose as well that $c_j(W) > 1$. If $p(b_j, b_{-j}), c_j(b_j)$ have real analytic extensions to an open neighborhood of $[\underline{w}_1, W] \times [0, W]$ then every equilibrium has finite support.*

Proof. Take $\underline{w}_{-1} = 0$ and consider

$$U_j(b_j) \equiv \int_{\underline{w}_j}^W p(b_j, b_{-j}) dF_{-j}(b_{-j}) - c_j(b_j).$$

We first show that this is real analytic in an open neighborhood of $[\underline{w}_j, W]$.

For c_j this is true by assumption so we show it for the integral

$$P_j(b_j) \equiv \int_{\underline{w}_j}^W p(b_j, b_{-j}) dF_{-j}(b_{-j}).$$

Let \mathcal{W} be the open neighborhood of $[\underline{w}_1, W] \times [\underline{w}_{-1}, W]$ in which p is real analytic. Then for each point $b \in \mathcal{W}$ the function p has an infinite power series representation with a positive radius of convergence r_1, r_{-1} for b_1, b_{-1} respectively. Hence the extension of p to a function of two complex variables has the same radius of convergence there. Take an open square around b_j in the complex plane small enough to be entirely contained in the circle of radius $\min\{r_1, r_{-1}\}$ and lying inside of \mathcal{W} . The product of these squares is an open cover of the compact set $[\underline{w}_1, W] \times [\underline{w}_{-1}, W]$, hence has a finite sub-cover. Choose the smallest square from this finite set, say with length $2h$. Then $p(b_j, b_{-j})$ is complex analytic in the domain $(\underline{w}_1 - h, W + h) \times (-h, +h) \times ((\underline{w}_{-1} - h, W + h) \times (-h, +h))$. Hence we may extend $P_j(b_j)$ to a complex analytic function in the domain $(\underline{w}_j - h, W + h) \times (-h, +h)$. This is a convex domain, take a triangular path Δ in this domain and integrate

$$\oint_{\Delta} P_j(b_j) = \oint_{\Delta} \int_{\underline{w}_j}^W p(b_j, b_{-j}) dF_{-j}(b_{-j}).$$

Everything in sight is bounded so we may apply Fubini's Theorem and interchange the order of integration to find

$$\oint_{\Delta} P_j(b_j) = \int_{\underline{w}_j}^W \left(\oint_{\Delta} p(b_j, b_{-j}) \right) dF_{-j}(b_{-j}).$$

By Cauchy's Integral Theorem since p is analytic $\oint_{\Delta} p(b_j, b_{-j}) = 0$. Hence $\oint_{\Delta} P_j(b_j) = 0$ so by Morera's Theorem $P_j(b_j)$ is analytic, and in particular real analytic when restricted to $(\underline{w}_j - h, W + h) \times 0$.

Hence the gain from deviating to b_j is given by a real analytic function $U_j(b_j) - \max_j U_j(\tilde{b}_j)$. That implies it is either identically zero or has finitely many zeroes. We can rule out the former case since $\max_j U_j(\tilde{b}_j) \leq 1$ and $c(W) > 1$. Hence F_j must place weight only on the finitely many zeroes. \square