UNIVERSITY OF CALIFORNIA

Los Angeles

LEARNING, EXPERIMENTATION, AND EQUILIBRIUM SELECTION IN GAMES

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
in Economics

by

Youngse Kim

1992

The dissertation of Youngse Kim is approved.

John G. Riley

Ivo Welch

David K. Levine, Committee Chair

University of California, Los Angeles

1992

DEDICATION

To Hyehoon Lee, my wife and colleague economist, and Sunghwan D. Kim.

Contents

List of Figures

List of Tables

ACKNOWLEDGEMENTS

I am delighted to acknowledge my thesis committee for their assistance. I wish to thank my committee chairman, David K. Levine, for his intellectual stimulus, valuable advice, and tireless encouragement. Joseph Ostroy and John G. Riley have remained my mentors throughout my academic life at UCLA. Discussions with Costas Azariadis, Sushil Bikhchandani, Kim Border, Bryan Ellickson, Drew Fudenberg, Jack Hirshleifer, Axel Leijonhufvud, Hamid Sabourian, Lloyd Shapley, Eric van Damme, Ivo Welch, and Bill Zame proved extremely beneficial.

I also thank Marco Celentani, In Ho Lee, and Wolfgang Pesendorfer, excellent members of the theory student seminar group—which has been led by Professor Levine—for intellectual and social comradery.

Financial support from the National Science Foundation, the Graduate Fellowship of the University of California, and the Academic Senate of UCLA.

Finally I acknowledge my parents, my wife, and my son for their love.

VITA

23 October 1962	Born, Seoul, Korea
1985	B.A., Economics Yonsei University Seoul, Korea
1987	M.A., Economics Yonsei University Seoul, Korea
March - August 1987	Lecturer School of Business Yonsei University Seoul, Korea
1988 - 1992	Teaching Assistant and Associate Department of Economics University of California Los Angeles, California

PUBLICATIONS AND PRESENTATIONS

Kim, Y. (October, 1990). "Characterization and Properties of Reputation Effects in Extensive Form Games."

______. (September, 1991). "Credibility, Smooth Expectations, and Inflation Persistence."

______. (July, 1992). "Adjustment Dynamics with Patient Players in Coordination Games." *UCLA Working Paper* no.667.

______. (October, 1992). "Adjustment, Evolution, and Equilibrium Selection in Games."

ABSTRACT OF THE DISSERTATION

LEARNING, EXPERIMENTATION, AND EQUILIBRIUM SELECTION IN GAMES

by

Youngse Kim

Doctor of Philosophy in Economics

University of California, Los Angeles, 1992

Professor David K. Levine, Chair

This dissertation addresses and resolves problem of selection among multiple equilibria in games, by perturbing the original system and then characterizing the outcome of resulting perturbed system.

Chapter 2 examines reputational sequential equilibrium of what we call participation games, that have many economic applicability, such as entry deterrence and product quality control. By perturbing the original game with types, we show that the lower bound of the single long run player's payoff is almost his Stackelberg commitment payoff in the limit as the finite horizon grows. Discontinuity exists between the infinite horizon and the limiting finite horizon solution. Our result is robust to a model modification in which the long run player announces the payoff structure just before the whole game begins so that the rational type of long run player has to mimic not only the strategy but also the initial payoff announcement of the Stackelberg commitment type.

Chapter 3 and chapter 4 analyze a game played by randomly and anonymously matched players from a large population. The game of interest is a multiperson

coordination game with multiple strict Nash equilibria. In chapter 3, players are fully rational, but adjustment is costly. Equilibrium outcomes are fully characterized as a function of group size and level of friction. We examine limiting results and their links to static equilibrium concepts. The equivalence between risk dominance and learning outcomes previously shown in two-player games fails with three or more players. Surprisingly, the limit as the friction disappears coincides with the selection from global perturbation and strict iterated admissibility. For a pure coordination game, a much stronger result can be shown to support equilibrium Pareto efficiency—as long as the friction is sufficiently small—regardless of group size either. Finally, we also provide numerical results that have some implications for several well-known experiments on coordination failure and history dependence.

Chapter 4 clarifies the relationship between adjustment or evolutionary dynamics studied in the literature. Two types of dynamic process turn out to possess the power of resolving indeterminacy: the deterministic adjustment dynamics with patient players, and the stochastic evolutionary dynamics with myopic players. Roughly speaking, the dynamically attractive outcome obtained with patient players corresponds to the static equilibrium assuming correlated play of opponents, while the long run state obtained with noisy myopic players corresponds to the static equilibrium selection predicted by independent play of one's opponents. We show that, if and only if group size equals two (ie, 2×2 games), the dynamic outcome from either type of process happens to coincide risk dominance. For any pure coordination game, a much stronger result obtains supporting the Pareto efficiency, regardless of the underlying dynamics.

Chapter 1

Overview

Chapter 2 examines reputation sequential equilibria of what we call participation games, that have many economic examples, such as entry deterrence and product quality control. By perturbing the original game with types. we show that the lower bound of the single long run player's payoff is almost his Stackelberg commitment payoff in the limit as the finite horizon grows. Discontinuity exists between the infinite horizon and the limiting finite horizon problem. Double traps may exist in some important subclass of games, namely enttry- inducement games. Our result is robust to a model modification in which the long run player announces the payoff structure just before the whole game begins so that the rational type of long run player has to mimic not only the strategy but also the initial payoff announcement of the Stackelberg commitment type. This chapter has a direct implications for some laboratory experimental results, such as Camerer and Weigelt (1988).

Chapter 3 and Chapter 4 focuse on games played by randomly and anonymously matched players from a large population. The class of games I study are symmetric, multiperson coordination games with multiple strict Nash equilibria. Existing refinements are powerless to choose between these equilibria. One way to resolve this indeterminacy is to consider an actual adjustment or learning process which operates in real time. If this process settles down to a unique outcome, then this outcome should be the analyst's prediction of how the game might be played. Therefore, this approach has the potential to explain how equilibrium is attained, and of singling out a unique equilibrium in situations where the underlying stage game has a plethora of outcomes.

Chapter 3, "Adjustment Dynamics with Patient Players," studies fully rational deterministic adjustment dynamics, in which players can revise their choices only periodically. Dynamic equilibrium outcomes are fully characterized as a function of the payoff matrix and the effective discount rate. More importantly, the dynamic outcome in the limit as players become very patient selects uniquely from the strict Nash equilibria depending on the payoff matrix. The resulting equilibrium selection suggests the following introspection arguments. Consider a player about to play a one shot n-player coordination game, in which there are two strict Nash equilibria where either all players choose action H, or else all players choose action L. No player knows which equilibrium will actually be played in advance. Each player faces $(n-1)$ opponents, and so there are n possibilities where in each possibility k out of $(n-1)$ opponents choose action H for $k = 0, \ldots, n-1$. Assume that the player places uniform probability $1/n$ on each of these possibilities, and on the basis of this assumption, the player is asked to choose his action. Notice that this probability assignment necessarily implies some degree of correlation between opponents' choices.

Surprisingly, the outcome just described coincides with static notion of equilibrium selection called global perturbation. Trembles are introduced into the game in such a way that payoffs are almost but not perfectly common knowledge among players, and that there is a chance that each of the actions can be a dominated strategy. More precisely, each player receives a noisy, private signal about the payoffs, but the player is unable to fully disentangle the true payoff realization from his private signal. Lack of common knowledge among the players makes it possible for strictly dominated strategies to exert an influence. This fact suggests that to solve the resulting incomplete information game we must use iterative elimination of strictly dominated strategies. The result of iterative strict dominance prescribes that *all players* play either one of the two actions, depending on the payoff matrix of the unperturbed game. Equilibrium selection based upon global perturbation refers to the one obtained at the exactly original game as common knowledge about payoffs becomes arbitrarily perfect.

On the other hand, in the opposite limit as players become myopic, both strict equilibria can be reached, and exactly which equilibrium will occur in the long run depends crucially upon the historical accident of the initial state. This is reflected in the fact that, in the framework of an evolutionary process which assumes myopia, Darwinian deterministic dynamics may well possess multiple steady states, and that the asymptotic behavior of the system depends on initial conditions. Trouble persists even if we perturb the deterministic dynamic system with a one-time mutation, which is the idea behind the concept of standard evolutionary stability. Moreover, notice that the connection between myopic replicator dynamics and strategic stability or rationalizability is vacuous in coordination games, since all strict Nash equilibria simply survive strict iterative admissibility.

Chapter 4, "Evolutionary Learning with Experimentations," resolves the equilibrium selection indeterminacy by introducing a probabilistic flow of small mutations or experimentations, thus making the dynamic system stochastic. The resulting stochastic law of motion possesses a well-defined, steady-state ergodic distribution. Consequently, this approach highlights certain strategy configurations as likely to be observed much more frequently than others, especially in the limit as the chance of mutations vanishes. It turns out that the power of distinguishing between multiple strict Nash equilibria returns even under myopia. The long-run state derived using stochastic evolutionary dynamics with myopic players corresponds to the static equilibrium selection motivated by the following introspection arguments. Consider again a player whose opponents choose either actions H or L, except that they choose with probability half on either action. Assume further that opponents' choices are done independently. Under this assumption, the player can calculate the expected payoff from each action, and the player should choose his own action on the basis of this calculation. Notice that this independence assumption contrasts sharply with the presumed correlation obtained with patient players. Also notice that, in two person games, this distinction simply disappears because each player has only one opponent.

Much of the existing literature has asserted that the limit dynamic equilibrium outcome coincides with Harsanyi and Selten's notion of risk dominance. In my two papers, I provide a overview of the connection between the nature of the adjustment dynamic process and static equilibrium selection. Generally, I refute the conjectured equivalence between the limit dynamic outcome and risk dominance. I also show that in two-person, bimatrix coordination games all the following equilibrium selection rules coincide with each other: (1) dynamic outcomes with patient players, (2) stochastic evolutionary dynamic outcomes, (3) static selection based on global perturbation, and (4) risk dominance. And finally, in general, for any pure coordination game, a much stronger result can be obtained supporting Pareto efficiency, irrespective of the underlying dynamics.

To recapitulate, the contribution of my research is threefold. First, my research provides a full characterization of the limiting dynamic equilibrium outcomes in multiperson games. Second, it clarifies the relationship between the nature of the dynamic system and static equilibrium selection. Roughly speaking, the limit dynamic outcome obtained with patient players corresponds to static equilibrium selection assuming correlated play by one's opponents, whereas the steady state with noisy myopic players corresponds to selection predicted by independent play by one's opponents. Third, it refutes the conjectured equivalence between limit dynamic outcomes and risk dominance. More specifically, this equivalence turns out to be only an artifact observable in two-person games.

Chapter 2

Reputation in Participation Games

2.1 Introduction

Consider a situation in which a single long run player faces a finite sequence of short run opponents, each of whom plays only once, but who observes all previous outcomes. The purpose of this paper is to see whether the long run palyer can acquire or maintain a reputation in participation games. In each stage participation game, a short run player decides first whether to choose an outside option or to enter a certain martket. With a short run player's staying out of the market, the stage game immediately ends. With her decision to participate into the relevant market, a chance is given for the long run player to move. The reader should notice that a distinguished feature of participation games lies in its sequential move structure. Depending on the stage game payoffs, participation games are divided into two interesting classes, entry-deterrence game and entry-inducement game.

A representative example of entry-deterrence games is Selten [1977] chain-store paradox. As a justification for seemingly irrational entry deterring behavior by means of irreversible investments and limit pricing, Kreps and Wilson [1982b] and Milgrom and Roberts [1982] study repatational equilibria in a framework of chain-store game. They showed that, if there is a small uncertainty about the payoffs of the long-lived incumbent(a là Kreps and Wilson), or if every player knows the payoffs of the incumbent but this is not common knowledge(a là Milgrom and Roberts), then reputation effects for predation would come into play. Such a reputation drives short run potential entrants out of the relevant market possibly except near the end of the game.

There are many practical applications which can be modelled as entry-inducement games, such as a quality game between consumers and a monopolistic producer (Fudenberg and Levine [1989a]), an asset market game between investors or workers and a capitalist (Fudenberg and Levine [1989b]), a sovereign debt game between foreign banks and a less developed country(Bulow and Rogoff [1989a,b]), etc. Consider a quality game, which is depicted in Figure 1. Call the "Stackelberg outcome" as the long run player's most preferred pure strategy profile of the stage game under the constraint that the short run player chooses a best reponse based on her beliefs and the strategies of her opponent. Then the Stackelberg outcome would be the monopolist's promising high quality and the consumer's purchasing, while the only perfect eqiulibrium is the consumer's not buying. On the contrary to the entry-deterrence game, the recognition that the long-lived producer has no method of demonstrationg that he is of high quality type so that building a reputation would be impossible has been pervasive. Indeed in the infinite horizon problem, it is not hard to contruct a pertrubation and an equilibrium of the resulting perturbed game such that the lower bound of long run player's payoff is less than Stackelberg outcome. This is no more the case in the limit of the finite horizon problem. The long run player even in the entry-inducement game can maintain a reputation so as to gain at least his Stackelberg payoff, although the reputational equilibrium is more fragile near the end game than thereof a deterrence game. With respect to this point, I show that double-sided traps exist in some final periods in an inducement game, whereas only one-sided traps do in a deterrence game. These observations are consistent with the experimental results as of Camerer and Weigelt [1988].

It may be interesting to enumerate other possible recipes to market failures which naturally occur in inducement games. First, precommitment or enforceable preplay contract will simply guarantee the long run player at least the Stackelberg payoff. Second, as Fudenberg and Levine [1989a] proposed, we may solve the transformed game with simultaneous move structure by requiring short run players to choose her best reponse from observationally equivalent set of actions. Their idea is based on the known fact that the long run player always can aquire a reputation for some commitment type in any simultaneous move game. Third, the long-lived guy may send some credible signals, such as advertising (Klein and Leffler [1981]) and private disclosure or warranty in a quality game (Grossman [1981]) and collateral in a debt game. The present paper does without any of the assumptions discussed above. Instead, I analyze whether a sequential reputational equilibrium can be constructed only by introducing small perturbations into original participation games.

This paper complements previous literature on reputational effects in simultaneous move games with long and short run players, which is mainly attributed to works by Fudenberg and Levine. Fudenberg and Levine [1989a] showed that if only pure strategies are allowed on the part of the single long run player, then he can obtain the Stackelberg commitment payoff except at most a fixed finite number of periods. Fudenberg and Levine [1992a] considered a situation in which mixed strategies are also allowed. According to their simple calculating method, the lower bound on the long run player's discounted normalized payoff would be very close to his Stackelberg commitment payoff. I derive similar results in participation games, which is a simple deterministic stage games but has a lot of practical applicability. While Fudenberg and Levine focused on Nash equilibria in infinite horizon problems, I characterize sequential equilbrium in a finite horizon problem and take its limit.

The paper consists of three parts. Section 2.2 analyzes the sequential equilibria in the simplest version of participation games, that is enough to contain all economic implications. Section 2.3 deals with the situation where the long run player determines the payoff structure before the whole game begins. This modification not only makes closer to the real world practice but also strengthens our result. Last section concludes.

2.2 The Model

There is a finite sequence of dates indexed backwards by t = T ,..., 2, 1. At each date t, there are two players, L and S_t. The single long run player L lives forever and a short run player S_t lives only one period during the date t. At the beginning of each date t, the entire past history of outcomes up to date t+1 is public information. We assume no discounting so that player L's time-averaging profit will be $\frac{1}{T}\sum_{t=1}^{T}\Pi_t$. A participation game is defined as in the opening section. Without any loss of economic intuition, consider the simplest case in which, given player S_t's decision to participate into the relevant market, player L must make a binary decision of whether to say yes or no. The general form of the stage game payoffs is depicted in Figure 2. We may assume that $b_y > b_0$ always holds.

I study only two versions of game of great importance: with $b_y > 0 > b_0$ in common, either $0 > a_y > a_0$, or $a_0 > a_y > 0$. All other cases are of little interest, since the unique perfect equilibrium will be trivial, no matter what one sided incomplete information in my sense there might be. I name the game with the first type of payoff structure as an entry-deterrence game (D-game for short) and the second as an entry-inducement game (I-game for short). A stage D-game has two Nash equilibria [Out] and [In, Yes], but only the latter one is subgame perfect.[1] On the other hand, a one stage I-game has the unique Nash and perfect equilibrium [Out]. Notice that, under the assumption of complete information, there is no reason why the equilibrium of the T period repeated game becomes anything other than the mere repetitions of the perfect equilibrium of the stage game.

[1]The other Nash equilibrium is subgame imperfect, since it can be supported only by an incredible off-the-equilibrium threat, i.e. player L's no.

Thus, a D-game refers to the situations where, even though the noncooperative equilibrium that naturally arises would be short run players' participations, the long run player wants them to stay out of the market. An I-game refers to the opposite situation. If we define a *Stackelberg payoff* as what the single patient player prefers most as far as short run opponents choose best response to their own beliefs and the long run player's strategies, it would be the value a in any participation game. The question is whether the single patient player can build the reputation in the I-game as nicely as in the D-game. The answer is positive. If and only if we let the original game perturbed by introducing a little incomplete infomation, we can construct a sequential reputational equilibrium in both D- and I-game. Moreover, this has a uniqueness property.

Throughout this paper, one sided incomplete information and perfect recall will be assumed. Also assumed is that any player in the game may implement neither precommitment technology nor signalling device. A single long run player and T short run players will play one of two possible games,[2] each of which involves T repetitions of a particuar stage game. This one-sided informational incompleteness stems from short run players' uncertainty about exactly which type of the long run player they are against. The long run player knows exactly which of these actually obtains. The first possible game is the original game, while the second one is the game in which the long run player behaves as if he committed himself to a particular action. The long run player in the original unperturbed game is called a "rational" type. He is called a "strong" and an "honest" type in the D-game and the I-game, repectively. Every short run player has an identical initial belief that the long run player is likely to be rational with probability $1-\rho$ and to be strong or honest with its complementary probability ρ, where $0 < \rho < 1$.

[2]Milgrom and Roberts [1982] analyzes a richer model with three types in the framework of D-game. We will consider two type case only at the expense of analytical complications.

A useful solution concept to analyze these games with incomplete information is the sequential equilibrium as developed in Kreps and Wilson [1982a].

Without loss of generality, we may normalize the payoffs as follows: let $a_y = 0, a_0 = -1, b_y = b, b_0 = b-1$, and L's payoff with S's Out equals a in the D-game and let $a_y = a, a_0 = 1+a, b_y = 1-b, b_0 = -b$ in the I-game, where $a > 0$ [3] and $0 > b > 1$.

Attention ought to be made on the I-game, thus all proofs and explanations will be made with repect to the I-game. For the purpose of comparisons, however, we also put down the results for the D-game in parenthesis. Let $p_T = \rho \in (0,1)$, and for $t = T-1, ..., 1$,

$$p_t = \Pr\{\text{L is of a honest(strong) type} \mid H_{t+1}^T\},$$

with the recursive definitions as follows:

i) S_{t+1}'s Out conveys no information, thus $p_t = p_{t+1}$.

ii) S_{t+1}'s In and L's Yes(No) together with $p_{t+1} > 0$ result in $p_t = \max\{b^t, p_{t+1}\}$.

iii) Otherwise, $p_t = 0$.

Whereas the honest(strong) L is always to say Yes(No), the strategy of the rational L would be as follows:

For t=1, say No(Yes) surely.

For t ¿ 1,

i) if $p_t \geq b^{t-1}$ then say Yes(No) surely.

ii) if $p_t < b^{t-1}$ then say Yes(No) with probability $q_t = \frac{(1-b^{t-1})p_t}{(1-p_t)b^{t-1}}$

[3] However, we assume $a > 1$ in D-game. The case of $0 < a < 1$ would result in a qualitatively similar characterization as $a > 1$ possibly except in the endgame. For details, refer to Kreps and Wilson [1982b] p.265.

Strategy of S_t is to choose In(Out) with probability:

1 if $p_t > b^t$.

$\frac{1}{1+a}\left(\frac{1}{a}\right)$ if $p_t = b^t$.

0 otherwise.

Proposition 1 *The strategies and belifs given above is a unique sequential equilibrium for the T-repeated I-game(D-game).*

Proof: The rough idea of the proof is as follows. Provided that the first period short run player S_t entered the relevant market and that the remaining periods were sufficiently long, even the rational L should behave as if he was of the honest type. The reason is that, if player L is somehow given an opportunity to move, to say no brings about an immediate gain of 1+a but zero in all subsequent dates since all the subsequent short run players interpreting L's previous saying no as a definite evidence that L is not an honest guy will simply stay out, whereas to say yes yields only a at the date T but a stream of positive expected profits later. Given L's stategy described above, S_t would participate into the market. In actuality, this is optimal for every S_t, $T \leq t \leq T^*$, and for the rational L during $T \leq t \leq T^* + 1$, where $T^* = \inf\{t \mid b^t < \rho\}$. Each player S_t from the date $T^* - 1$ on randomizes optimally, as long as all previous short run players actually came in and player L always responded with yes. It is in player L's interest to start randomization between yes and no from the date T^* on.

To instruct the reader, I provide a complete calculation of the equilibrium for the $T = 2$ case only. Moreover, I will not mention the behavior of the honest type, since he has no alternative but to always say yes whenever player S enters. For $T = 1$, it is trivial to show that S_1 would choose coming in, staying out, or randomizing iff $p_1 > b, p_1 < b$, or $p_1 = b$, respectively, and that the rational L responds with no, the stage game best response, with probability one. Now for $T = 2$, let us suppose that player L's strategy at t=2 is to say yes with probability q_2 and say no with probability $1 - q_2$. The optimality principle of Bellman implies that player L must be indifferent between the strategy prescribed above and that of surely saying no. That is, $(1-q_2)(1+a) + q_2(a + \Delta(1+a)) = 1 + a$, which, together with the consistency of beliefs on the sequential equilibrium path, gives rise to:

$$\Delta \equiv \Pr(S_1 \text{ comes In } \mid S_2's \text{ In was responded with Yes}, \; S_1 \text{ indifferent between In and Out}) = \frac{1}{1+a} \quad (2.1)$$

Let us look at the short run players' beliefs and behaviors. Given player S_1's observation of L's having said yes at t=2, his posterior probability that L is honest will be revised using Bayes' rule, so that we have

$$\begin{aligned} p_1 &= \Pr(\text{L is honest}|\text{In and Yes observed at t=2}) \\ &= \frac{p_2}{p_2 + q_2(1-p_2)} \end{aligned} \quad (2.2)$$

If and only if this posterior probability is greater than (resp. equal, less than) b, then S_1 should enter with prob 1 (resp. prob $\Delta = \frac{1}{1+a}$, prob 0).

By the consistency of beliefs required on the equilibrium path, q_2 is computed so as for $p_1 = b$ to hold, i.e.

$$q_2 = \frac{(1-b)p_2}{(1-p_2)b} \quad (2.3)$$

Player S_2 will enter, randomize with probability $\Delta = \frac{1}{1+a}$, or stay out, according as $[p_2 + q_2(1-p_2)](1-b) + (1-q_2)(1-p_2)(-b)$ is greater than, equal to, or smaller than zero, repectively, i.e. $p_2 >, =,$ or $< b^2$ using the eq (1). It can be easily checked that there are three possible situations depending upon the size of ρ and

b. In the first situation where $\rho < b^2 < b$, any type of player L gets nothing with probability one, since both S_1 and S_2 would stay out so even the honest long run player has no opportunity to demonstrate the truth. In the second situation in which $b^2 < b < \rho$, both S_1 and S_2 would surely enter and player L at t=2 would surely say yes but the rational L at t=1 surely says no on the equilibrium. In the last case of $b^2 < \rho < b$, player S_2 would come in with probability one, player L at t=2 randomizes, player S_1 randomizes as long as he L acually obseved payer L's saying yes at the previous date t=2.

For general $T > 2$, the reader can easily verify[4] not only Bayesian consistency of S_t's beliefs and L's strategies but also optimality of every player's moves starting from any information set of the game. Then the optimality principle of Bellman ensures that no player can benefit by unilaterally changing its strategy starting from any point. $\blacksquare$

The proposition above seems to show that the properties as well as the paths look identical on the sequential reputational equilibria for T-repeated games of both D- and I-game. However, their qualitative nature and economic implication are very much different mainly in that the reputational equilibrium is far more fragile in the I-game than in the D-game. For a clearer comparison, we should investigate their evolutionary structures. Let us look at the D-game at the date $T^* - 1$ where $p_{T^*-1} > 0$, which implies that all the previous S's participations have been met by player L's response of no. Now if S_{T^*-1}'s randomization leads him to staying out of the market, then his immediate successor S_{T^*-2} would certainly enter (since $p_{T^*-2} = p_{T^*-1} = b^{T^*-1} < b^{T^*-2}$) and player L at the date $T^* - 2$ would randomize. If player L happens to say no at the date $T^* - 2$, his reputation for

[4]Refer to details in Kreps and Wilson [1982b] p.259-260, or Milgrom and Roberts [1982] p.306-311.

toughness could be restored so that again $p_{T^*-2} = b^{T^*-2}$ attains. The game will evolve in the same manner for any $t = T^* - 1, ..., 3, 2$. In other words, the long run player can demonstrate that a short run player's decision of entering was mistaken even near the end of the D-game in a weak sense that he actually does this only in the course of optimal randomizations. On the contrary, there is a "double-sided trap" in each date after T^* in the I-game. The first trap refers to the situation where player L loses his reputation for honesty in the event of saying no, which stems from player L's randomization processes. The D-game also has this feature in common. More importantly and specially only in I-game, the following second trap comes from S's randomizations. Supposed that $p_{T^*-1} > 0$ and that S_{T^*-1}'s randomization leads her to staying out, then every subsequent short run player will simply stay out. This may happen with non-negligible probability although the long run player has been always replied with yeses. Moreover, once this happened, even the honest guy has no way of demonstrating his honesty. In summary, the I-game reputational equilibrium is far more fragile, in the sense that a player S's observing not only no by player L but also out by one of her predeccessors makes her simply choose staying out of the relevant market.

Immediate from the results thus far is the following:

Corollary 1 *Fix any participation game. In the limit as the horizon goes to infinity, the lower bound that the long run player obtains is almost his Stackelberg payoff.*

In the infinite horizon participation game, it is easy to construct a situation in which the long run player cannot obtain his Stackelberg payoff.[5] Hence, there is a discrepancy between the limit of the least equilibrium payoff to the long run player as its finite horizon goes to infinity and that when the horizon is infinite.

[5] One can find an example in Fudenberg and Levine [1989a].

2.3 Announcement and Commitment

A practical aspect that many examples of the I-game have in common may be that, given a short run player's participation into the relevant market, there is a tradeoff between the long run player's short term profit and the relevant short run player's payoff. Moreover, their payoffs are usually control variables the long run player can determine. In a quality game, given a consumer's decision to purchase one unit of goods the monoplist wants to sell, a negative relationship between the level of product quality and the monopolist's short term profit seems to obviously exist. In an asset market game as in Fudenberg and Levine [1989b], after some investors or workers provide their assets or labors to the single patient capitalist, a similar conflict may exist between returns to investors wage compensations to the workers and profits to the capitalist.

To investigate this situation, we slightly modify the payoff structure. As before, player S_t's choosing an outside option yields nothing to both player L and himself. Player S_t's participation directly brings about -1 to himself and y to player L. Here -1 that player S_t gets can be interpreted as disutility from consuming low quality goods in a quality game and as value of financial assets provided to the capitalist in an asset market game. The long run player decides whether to offer a compensation $1+w$ to the short run player or not at all. We assume that player L determines a level of w and that all the short run players somehow get to know the precise value of w before the whole game begins.[6] Presumably, a condition that $1+w>0$ must hold, since otherwise In is a dominated strategy for player $S_t, \forall t$, thus every S_t will simply stay out. On the other hand, player L has no incentive to offer the gross compensation greater than y, so that $y>1+w$ also

[6]This is not an innocuous assumption. Refer to Hart and Tirole [1988] for some results without this restriction.

holds. Some reader might guess that player L has no incentive to offer more than $1 + \epsilon, \forall \epsilon > 0$. This is wrong because a reduction of the compensation by player L brings about not only benefits from directly raising his own share but also costs from losing some customers who would have surely come in before.

As a prelimenary for the main result of this section, the reader can check the following lemma by mimicking proofs of proposition 1:

Lemma 1 *For w fixed, the beliefs and strategies described below is the unique plausible sequential equilibrium for a perturbed T-repeated game.*

Beliefs of S_t.

i) S_{t+1}'s staying out reveals no information, thus $p_t = p_{t+1}$,

ii) S_{t+1}'s In and L's Yes together with $p_{t+1} > 0$ result in $p_t = \max\{(1+w)^{-t}, p_{t+1}\}$,

iii) Otherwise, $p_t = 0$.

Strategy of the rational L.

For $t = 1$, say No surely

For $t > 1$,

i) if $p_t \geq (1+w)^{t-1}$, then say Yes surely,

ii)if $p_t < (1+w)^{-t+1}$, then say Yes with prob $q_t = \frac{p_t}{1-p_t}[(1+w)^{-t+1} - 1]$.

Strategy of S_t.

In surely if $p_t > (1+w)^{-t}$,

In with prob $\frac{1+w}{y}$ if $p_t = (1+w)^{-t}$,

Out surely otherwise.

Let us define $T^* = \inf\{t \mid (1+w)^{-t} < \rho\}$. On the sequential equilibrium path, every short run player S_t for $t = T, T-1, ..., T^*$ participate into the market with probability one, and player L optimally replies with sure yeses to those entries up to t $= T^* + 1$ and then randomizes thereafter. Now suppose that player L can determine w before the whole game begins. Let w^* be the level of net compensation that maximizes player L's time-averaging payoff in a T-repeated game. We should notice that player L may lose some sure customers by raising his own share $(y - (1 + w^*))$, thus there is a tradeoff between w^* and T^*. Notwithstanding, it is optimal for player L to reduce the value of w^* as much as he can keep the number of short run players who surely enter the same as before. Therefore, the profit maximization of the rational long run player requires the local condition, which states formally: For any type of player L and for given T^*, profit maximizing w^* must satisfy $(1 + w^*)^{-T^*} = \rho$.

First, we calculate the best randomizing strategy on the part of the rational L. Since his time-averaging payoff is $V_R \equiv \frac{1}{T}[(T - T^*)(y - (1 + w^*)) + y]$ by using the optimality principle of Bellman, the rational player L's objective will be to maximize V_R subject to

(LOC) $(1 + w^*)^{-T^*} = \rho$,

(ICC) $y > 1 + w^* > 1$,

given $y > 1, \rho > 0,$ and T. Define a pair (w_R^*, T_R^*) to be the rational L's maximization solution.

Now we characterize the optimal announcement on the part of the honest type of the long run player. Recall that the sequential equilibrium of I-game suffers from double traps in the endgame. ¿From player S_t's strategy described in Lemma 1 and the local condition for profit maximization, the honest type's expected payoff along the equilibrium path would be

$$V_H = \frac{1}{T}[y-(1+w^*)][T-T^*+1+\Delta+...+\Delta^{T^*-1}],$$

where $\Delta = \frac{1+w^*}{y}$ and $(1+w^*)^{-T^*} = \rho$. Rearrangement yields

$$\begin{aligned} V_H &= \frac{1}{T}[[y-(1+w^*)](T-T^*)+y-\frac{1}{\rho y^{T^*-1}}] \\ &= V_R - \frac{1}{T\rho y^{T^*-1}} \end{aligned} \tag{2.4}$$

It is not difficult to check that $\frac{\partial V_H}{\partial T^*} > \frac{\partial V_R}{\partial T^*} = 0$ and $\frac{\partial^2 V_H}{\partial T^{*2}} < \frac{\partial^2 V_H}{\partial T^{*2}} < 0$ at $T^* = T^*_R$, unless y is too small. Henceforth, if we denote the honest type's optimal randomizing strategy as (w^*_H, T^*_H), it is true that $w^*_H < w^*_R$ and $T^*_H > T^*_R$. This implies that,in order to conceal his type, the rational L has to propose the same payoff announcement as the honest type, so that he offers a smaller compensation to short run players and has to sacrifice some of the sure customers.

The point is that the rational L must mimic the behavior of the honest counterpart in terms of not only actions but also payoff announcement. Even with this additional constraint, we get the following:

Proposition 2 *In the limit as T goes to infinity, we have*
i) $T^* \to \infty$, but $\frac{T^*}{T} \to 0$; ii) $w^* \to 0$. Moreover, $\frac{\partial T^*}{\partial y} < 0$, $\frac{\partial T^*}{\partial \rho} > 0$; $\frac{\partial w^*}{\partial y} > 0$, $\frac{\partial w^*}{\partial \rho} < 0$.

Proof: I deal with T^* as continuous variable, since doing so loses nothing but calculating complications. Applying the Lagrangian method to the maximization problem together with (LOC) to substitute w^* for T^* and rearranging the resulting equation, we have

$$T = T^* + \frac{\rho^{\frac{1}{T^*}} y - 1 - \frac{\log y}{\rho y^{T^*-1}}}{-\log \rho} T^{*2} \tag{2.5}$$

Given $y > 1$ and $0 < \rho < 1$, the condition that $T \to \infty$ requires $T^* \to \infty$. Now it is clear that

$$\frac{T^*}{T} = [1 + \frac{\rho^{\frac{1}{T^*}} y - 1 - \frac{\log y}{\rho y^{T^*-1}}}{-\log \rho} T^*]^{-1} \tag{2.6}$$

thus

$$\lim_{T \to \infty} \frac{T^*}{T} = \lim_{T^* \to \infty} \frac{T^*}{T} = 0,$$

together with $\lim_{T \to \infty} T^* = \infty$. We proved i).

On the other hand, taking log to both sides of (LOC) and rearranging yields $w^* = \rho^{\frac{-1}{T^*}} - 1$, thus $\lim_{T \to \infty} w^* = \lim_{T^* \to \infty} w^* = 0$. The second part ii) is also done. ∎

The proposition above implies that, as the horizon gets larger and larger, the number of short run players who optimally randomize near the end of the game also should be controlled larger, while its relative proportion gets negligible. In other words, the proportion of sure customers who enter the market with probability one monotonically approaches to unity. In addition, the long run player can optimally reduce the amount of net compensation that provides to some short run players incentives to participate into the relevant market. The second part shows some comparative statics which states that the optimal net compensation become smaller as the horizon gets larger, as shortrun player's probability assessment that player L is of the honest type gets bigger, and as the total revenue to player L gets smaller. As a consequence, the long run player is able to obtain almost extensive form Stackelberg payoff for sufficiently long horizon T. Moreover, in the limit as the horizon T approaches to infinity, the ϵ-first-best is indeed attainable.

2.4 Final Remarks

Consider repeated games in which a single patient player plays against a finite sequence of short run opponents. As a particular deterministic stage game in which short run players move first in each stage game, a participation game may have many practical applications, such as entry-deterrence behavior by an incumbent, a quality choice by a monopolist, a debt decision by a less developed country, etc.

I study charaterizations and properties of *the* sequential equilibrium in finitely-repeated participation games only by introducing a small perturbations into the original game. I show that reputation effects play an important role in any participation game as almost nicely as in any simultaneous move game. This is a surprising counterargument to the common view that the single patient player is not able to acquire or maintain a reputation in finitely repeated I-games. As a limit of finitly repeated participation games, the reputational equilibrium of the infinitely repeated game is also well characterized. Therefore, the fact that the long run player can obtain his Stackelberg payoff was shown, although the Stackelberg payoff here is differently calculated from that of simultaneous move games. These consequences are robust to a model modification where the long run player announces the payoff structure, which puts an additional contraint on the behavior of rational type.

An important problem that is worth being pursued will be to calculate the lower bound that the long run player can obtain on any sequential equilibrium of general extensive form game in the limit as the horizon grows.[7]

[7] Schmidt [1990, 1992] analyzes repeated bargaining problem and games with conflicting interest.

Figure 2.1: Product Quality Game

Figure 2.2: Entry Deterrence Game

Figure 2.3: Participation Game

Chapter 3

Adjustment Dynamics with Patient Players

3.1 Introduction

In this paper, we analyze a game played by randomly and anonymously matched players from a large population. Players face a perfect foresight deterministic dynamic process with costly adjustment. The class of games we study are symmetric binary action mutiperson coordination games with two strict Pareto-ranked Nash equilibria.[1] Consideration of these games is motivated by the simple game-theoretic issue of selection in games with multiple equilibria in which existing refinements are powerless. For instance, many of the stringent solution concepts proposed in the literature, such as the strategic stability of Kohlberg and Mertens [1986], are silent concerning the selection among several strict Nash equilibria. Furthermore, some recent studies on learning and evolution have also addressed the question of how a particular equilibrium will emerge in a dynamic context.[2] Although some convergence results are obtained, these studies do not offer an equilibrium selection criterion, since in these models all strict Nash equilibria share the same dynamic properties.

One approach for resolving this indeterminacy is to consider an actual adjustment process which operates in real time, and to see what limit outcomes if any might appear. For example, we allow players to have the opportunity from time to time to revise their choices given what their opponents are currently doing, and given the "correct" expectation about the future play of the game (namely, perfect foresight). If this continuous revision process settles down to a unique outcome, then this outcome should be the analyst's prediction of how the game

[1]This class of games represents, in a stylized fashion, the types of interactions prevalent in network externalities such as compatibility of computer software, video tapes, typewriter keyboards, and language, as well as many recent Keynesian macroeconomic models of coordination failures, geographical formation of core and periphery (Krugman [1991]).

[2]A partial list of this literature includes Jordan [1991], Fudenberg and Kreps [1992a], Canning [1992], Milgrom and Roberts [1990, 1991], and Fudenberg and Levine [1992a, b].

might be played. Therefore, this approach has the potential of explaining how an equilibrium is attained, and of singling out a unique equilibrium in situations where the underlying static game has multiple Nash equilibria.

This avenue has been explored in recent adaptive or evolutionary formulations, [3] most of which have asserted that the limit dynamic equilibrium outcome coincide with Harsanyi and Selten's [1988] static notion of risk dominance. The limit behavior of Blume's [1] dynamic process with respect to parametric changes that make strategy revisions a best response is shown to give rise to the same outcome as risk dominance selection in coordination games. Kandori, Mailath, and Rob [1992] consider evolutionary models for a finite population in discrete time with constant flow of mutations, which generate Markov processes in the behavioral pattern. Fudenberg and Harris [1992] study a version of the replicator dynamic in continuous time for a large population. In this paper, the random perturbation of the system is introduced by a Brownian motion. These last two papers show the same result: for 2×2 games, as the mutation rate and noise go to zero, the distribution becomes concentrated on the risk dominant equilibrium. Lastly, Matsui and Matsuyama [1991]—from which the present paper borrows heavily—shows an equivalence between risk dominance and dynamic stability in a two person bimatrix game of common interests.

The results of this paper cast strong doubt on the conjectured equivalence between the limit dynamic outcome and risk dominance. We will show that, for the Matsui and Matsuyama approach, these two notions "happen" to coincide only in

[3] This is nothing but one strand of numerous frameworks. Other popular and interesting approaches, which specifically study games with multiple equilibria, are fictitious learning (Krishna [1992]), learning with bounded memory or finite automata (Aumann and Sorin [1989]; Binmore and Samuelson [1992]), Turing machine learning under computability (Anderlini and Sabourian [1991] and references therein), and so on. Another game of great importance is prisoner's dilemma, which Young and Foster [1991] analyzes using stochastic evolutionary dynamics, and Nowak and May [1992], Glance and Huberman [1992], and references therein provide computer simulation results in machine learning framework.

the two person bimatrix game. First, we will fully characterize the dynamic equilibrium outcome in terms of group size and a friction parameter which depends positively on the discount rate of players and negatively on the chance of their action switches. To say the friction disappears implies that players are very patient, or that each player can revise his choice whenever he wants. On the other hand, to say the friction grows without bound implies that players are myopic, or that they choose strategies once and for all. A state is said to be globally attractive if there exists an equilibrium path that reaches or converges to that state from any initial condition. It is shown that, in the limit as the friction vanishes, either everyone's playing one action or everyone's playing the other will be the globally attractive state, depending upon the payoff matrix.

Surprisingly, the limit as the friction approaches zero turns out to coincide with Carlsson and Van Damme's [1990, 1991] notion of equilibrium selection through perturbation of the original game. The original game is perturbed in such a way that each player receives a private signal about the payoffs, but is unable to fully disentangle the true payoff realization from one's private noise. Lack of common knowledge among players makes it possible for strictly dominated strategies to exert an influence. This fact suggests that, to solve the resulting incomplete information game, we must use iterative elimination of strictly dominated strategies. When common knowledge about payoffs becomes arbitrarily perfect among players, the result of iterative strict dominance prescribes that all players play either one or the other action, depending on the payoff matrix of the original unperturbed game. The major argument of this paper is that this globally attractive dynamic outcome, in the limit as the friction disappears, coincides with equilibrium selection based on a global perturbation approach.

As verified by Matsui [1992], the opposite limit as players become myopic is closely related to a version of evolutionary stability, attributed to Swinkels [1992a]. Note that the evolutionarily stable strategy is defined as a strategy distribution which is robust against a once-and-for-all invasion by a small number of mutants. For this limit dynamic, the payoff matrix does not matter at all and only the initial fraction of each action type does. Such indetermincy is resolved if we perturb the dynamical system with a constant flow of mutations and experimentations. The idea behind mutations is to test the stability of states by repeatedly subjecting them to disturbances, and observing to which states the society tends to return. My companion paper [19] not only characterizes the long run ergodic distribution in the limit as the probability of mutations vanishes, which suggests a criterion for selecting among multiple strict Nash equilibria; it also clarifies the link between the features of the underlying dynamics and the static equilibrium selection. Roughly speaking, the long run state obtained with patient players corresponds to the static equilibrium assuming correlated play of opponents, while the long run state obtained with myopic players corresponds to the static equilibrium selection predicted by independent play of one's opponents.

The present paper may also have substantial implications with regard to recently developed experimental results by Van Huyck et al. [1990, 1991] on coordination failure, and by Cooper et al. [1990] on the predictability of Nash equilibrium. In particular, we provide theoretical and numerical evidence that is consistent with the following observations:

- weak dominance,
- a wide dispersion of initial effort choices,
- a trend to drift in small group treatments,
- a rapid convergence to the Pareto-worst Nash equilibrium regardless of initial strategic uncertainty when the group size is large and the summary

statistic affecting each player's payoff is the minimum effort choice among the group (i.e. large group minimum treatment),

- a strong history dependency in large group median treatments,
- and Pareto efficiency in pure coordination problems.

To recapitulate, coordination failure and history dependency are the most remarkable features, respectively, under minimum and median rule, when the group size is large.

The balance of the paper is organized as follows. Section 3.2 offers an intuitive exposition of the basic idea with a simple example. Section 3.3 formally defines the game of interest. Section 3.4 sets up the dynamic model and then characterizes its dynamic equilibrium outcomes. Section 3.5 contains calculations two important static equilibrium selection concepts, namely global perturbation and risk dominance. This same section proves the equivalence between the limit adjustment dynamic outcome and the static equilibrium selection based on global perturbation. Section 3.6 proves a strong result that, in any –symmetric or asymmetric – 2×2 game, the limit dynamic equilibrium outcome coincides with the selection based on global perturbation. This is also true even if the speed of adjustments are different, as long as the proportion between them is fixed in the limit. Section 3.7 studies two interesting subclasses of the original game, that is, pure coordination and stag hunt games. Section 3.8 offers numerical evidence within the framework of the stag hunt game used in experimental studies. The last section concludes with some suggestions for future research.

3.2 An Exposition

Consider the following highly stylized game. A forest is inhabited by a stag and a number of hares. There are n identical hunters that simultaneously and without communication have to choose between hunting a stag or a hare. If a player decides to hunt for a hare, his payoff is $\$x$ no matter what the other hunters may choose. If the player decides to pursue a stag then his payoff is determined not only by his own choice but by the actions of others, summarized by a simple statistic. Roughly speaking, stag hunting is successful only when enough of the hunters cooperate. Under a "minimum rule," even a single defection from full cooperation results in a failure. Under a "median rule," the cooperation of 50% of the hunters is sufficient for a successful stag hunt. We further specify that a successful stag hunt yields \$10 to each of the participants, whereas a failure brings about nothing. The normal form of this game is called the *stag hunt game*, the two and three-player version of which are depicted in Figure 3.1 and Figure 3.2.

We first study the dynamic evolution of the social equilibrium played by a large population. Each hunter is randomly and repeatedly matched with $(n-1)$ other players to play the stage stag hunt game anonymously. Players are fully rational, maximizing their discounted average expected payoff, with the dynamic path on which they condition their expected payoffs perfectly foreseen. However, there is friction: not every player is able to switch his own action every period. While this assumption is stylized, it can be interpreted as a transaction cost. For example, it could be the cost of switching from rabbit traps to hunting rifles. All the hunters are assumed to observe what fraction of hunters in the society as a whole are choosing between hares and stags. Given the opportunity to switch, each hunter chooses an action that maximizes his expected utility conditional on the correct expectation of the future play of the game. The dynamic equilibrium outcome is fully characterized as a function of group size n, and the effective discount rate ρ. The effective discount rate takes into account both the real-time discount rate and the cost to switching actions. The long run steady state of the social equilibrium must end up with either everyone hunting stag or everyone hunting hares, depending upon the payoff x and the initial fraction of stag hunters. While not regretting their individual choice in either equilibria, people in hare hunting society are nevertheless worse off than those in stag hunting society. Defection by a single individual or a negligible number of agents is simply in vain. In other words, all hunters may be playing a best response, but there is a chance that these best responses implement a Pareto-inferior equilibrium.

To take a concrete example, let $n = 2$ and $\rho = \frac{1}{2}$. It can be shown that the "good" stag ("bad" hare) equilibrium can result regardless of the initial population fraction of hare hunters, if the sure return to rabbit hunting x, is smaller than \$4 (greater than \$6).[4] In the case where x is between \$4 and \$6, the historical accident with regard to the initial fraction of hunter types plays a crucial role in determining exactly the long run equilibrium. Now, as players become more patient in the sense that ρ approaches to zero, the middle region of history dependency shrinks. There is a single limiting threshold value of x equal to the average payoff of the stag hunt under the assumption that the possibility of each opponent's hunting stag is equally likely. According to this assumption, since the probability that the opponent chooses stag and the probability that his opponent chooses hare are equal, the threshold is $\frac{1}{2} \times 10 + \frac{1}{2} \times 0 = \$5$. To compare with another set of parameters, let $n = 3$ and $\rho = \frac{1}{2}$, under a minimum rule. Then the history dependent region would be between \$2.28 and \$4.28, which shrinks to an infinitesimal area around $\frac{1}{3} \times 10 + \frac{2}{3} \times 0 \;=\; \$3.33$, if people care very much about their future. Roughly put, in the limit as people are extremely patient, the society eventually settles down on the stag (hare) equilibrium if x is smaller (greater) than \$3.33. For a last example with $n = 7$ under a median rule, it is

[4]The reader will have to trust me for the accuracy of these numbers, which are calculated using Eqs. (3.5)–(3.8) derived in Section 3.4.

easy to check the limit critical value equals $\frac{4}{7} \times 10 + \frac{3}{7} \times 0 = \$5.71$.

We now turn to Harsanyi and Selten's [1988] notion of risk dominance. The definition of risk dominance is based on a hypothetical process of expectation formation starting from an initial situation where it is common knowledge that either the stag equilibrium or the hare equilibrium must be the solution without knowing which one is the solution. Consider a process in which players first, on the basis of a preliminary theory, form priors on the strategies of their opponents. The preliminary theory can be summarized as follows: (i) Each player i believes that either all the other players hunt stag with a subjective probability z_i or all other players hunt hares with its complementary probability; (ii) each player i plays his best response to this belief; (iii) the z_i are independently and uniformly distributed on $[0, 1]$. Unfortunately this naive theory will not work since this best reply strategy combination will generally not be an equilibrium point of the game, and therefore it cannot be the outcome chosen by a rational outcome selection theory. The second-order best reply to the first-order vector is iteratively calculated. Thereafter, players gradually adapt their prior expectations to the final equilibrium expectations by means of a *tracing procedure*. As the tracing procedure progresses, both the prior vector and the best response strategy combination are subjected to systematic and continuous transformations until both of them finally converge to a specific equilibrium of the game. Thus at the end of the tracing procedure both the players' actual strategy plans and expectations about each other's strategy plans will correspond to the same equilibrium point, representing the risk dominant outcome.

Fortunately, the tracing procedure can be accomplished in one round in the present stag hunt game. Consider the $n = 2$ case. According to steps (i) and (ii) of the preliminary theory, hunter i chooses stag if:

$$z_i \times 10 + (1 - z_i) \times 0 \; > \; x \qquad \text{or} \qquad z_i \; > \; \frac{x}{10}$$

where $i = 1, 2$. Using step (iii), player i knows that his opponent chooses stag with probability $(1 - \frac{x}{10})$ and hare with the complementary probability $\frac{x}{10}$, therefore the prior must be revised to the posterior $(1 - \frac{x}{10})$. Now player i optimally hunts stag if

$$(1 - \frac{x}{10}) \times 10 + \frac{x}{10} \times 0 > x \qquad \text{or} \qquad x < \$5.$$

We conclude that the stag hunt risk dominates hare catching if $x < \$5$ and vice-versa. For another example, let $n = 3$ under a minimum rule. Following the steps described above suggests that the critical x value be the solution to $10 \times (1 - \frac{x}{10})^2 + 0 \times [1 - (1 - \frac{x}{10})^2] = x$, so that risk dominance selects stag if $x < \$3.82$.

Finally we examine Carlsson and Van Damme's [1990, 1991] notion of global perturbation. It is based on the idea that players are uncertain about the payoffs of the game. Trembling the game is made in such a way that payoffs are almost but not perfectly common knowledge, and that there is a chance that each of the actions can be a dominated strategy. To be specific, assume that the true number of hares is uncertain. One extreme possibility is that no rabbit might be available so that hare hunting only incurs an effort cost, while the other extreme possibility is that the forest might be indeed crowded with rabbits so that even a successful stag hunt fares worse than the hare hunt. More formally, there is a small but non-negligible probability that $x < 0$ in which rabbit hunting is strictly dominated, and $x > \$10$ in which stag hunting is strictly dominated. Each hunter receives a private signal that provides an unbised estimate of the true common value x, but the signals are noisy so the true value of x will not be common knowledge.

The player then chooses whether to hunt stag or hare. Assume that the noise can be at most \$0.50. For instance, if the true value of x is \$5.50, then all the private signals must be somewhere between \$5 and \$6 from an outsider's point of view. Imagine a situation in which a particular hunter i just observes his private signal x_i equal to \$5.50. Even if he knows, upon having observed \$5.50, that the true x lies between \$5 and \$6 and that all the other x_j's are between \$4.50 and \$6.50, this is in fact not common knowledge to hunters i and j. Now suppose that hunter j observes $x_j =$ \$4.50. Hunter j knows that the true x lies between \$4 and \$5, and x_i lies between \$3.50 and \$5.50. The problem is that hunter i does not know that hunter j knows that his x_i lies in the interval [\$3.50, \$5.50]. Lack of common knowledge expands all the way down, and therefore enables remote areas of dominated strategies where x is negative or greater than \$10 to exert an influence. This argument may well apply to all the other less extreme realizations of x_j in the interval [\$4.50, \$6.50], say \$5.70 instead of \$4.50, and any smaller size of the maximum noise, say a dime or a penny instead of 50 cents. Later we will show that equilibrium can be characterized using iterative elimination of strictly dominated strategies, and that it possesses a cutoff property. Finally, we are interested in what happens with the payoff realization that corresponds to the original game.

Take the $n = 2$ game. As was suggested, we maintain the assumption that no player will choose strictly dominated strategies. Hunter i will certainly choose stag if the secure return from catching hare is negative, i.e. $x < 0$. Since the expected true value of x conditional on his private signal x_i is simply x_i because of unbiasedness, player i knows that stag is strictly dominant at each such observation. Consider x_i to be slightly above zero. Notice that, with the additional assumption that the private signal is uniformly distributed around the true common value of

x is imposed, the probability that your x_j is bigger than my x_i does not depend upon x_i, thus the conditional probability must always be half. Player i knows that his opponent j will hunt stag if $x_j < 0$, hence i's payoff if he hunts stag at x_i would be approximately $\frac{1}{2} \times 10 + \frac{1}{2} \times 0 =$ \$5. As an astute reader might see, the process of eliminating strictly dominated strategies ends up in one iteration for stag hunt games. In other words, the same logic applies not only to an x_i slightly above zero, but also to any x_i below \$5. Since the expected payoff from hare hunting is x_i, we conclude that each hunter should hunt a stag (hare) if his private signal about the secure return form hunting hares is smaller (larger) than \$5. For example with $n = 3$ under a minimum rule, it can be similarly calculated that a hunter should choose a hare only when his private signal is smaller than $\frac{1}{3} \times 10 + \frac{2}{3} \times 0 =$ \$3.33.

Table 3.1 provides the cutoff values calculated for the limit adjustment dynamic outcome, risk dominance, and global perturbation in the case of minimum and median rules when the number of players are $n = 2, 3, 15, 99$.[5] The reader may be aware that the dynamic equilibrium outcome selection in the limit as the effective discount rate ρ goes to zero coincides with the static equilibrium selection based on global perturbation but not of risk dominace. This is not by chance! We will verify this equivalence in general coordination games.

3.3 The Game

We consider a symmetric n-person coordination game with binary actions, denoted High (H) and Low (L). The normal form game denoted by $G(n, \Pi)$ has 2^n number

[5]It is interesting to note that under the minimum rule, global perturbation is more conservative than risk dominance, in the sense that there is a subset of x such that risk dominance selects the risky Pareto-superior choice while global perturbation prescribes the secure Pareto-inferior action. This observation implies that coordination failures are more severe from the viewpoint of global perturbation than of risk dominance.

of cells, but due to symmetry only $2n$ cells need to be taken into account. Consider a strategy profile in which k agents choose H with the remaining $(n-k)$ agents choosing L; we denote π_k^H and π_{n-k}^L to be the payoff for a player taking H and L respectively, where $k = 1, 2, \ldots, n$. Let a vector $\Pi^\zeta = (\pi_1^\zeta, \pi_2^\zeta, \ldots, \pi_n^\zeta)$, for $\zeta = \{\text{H, L}\}$, and $\Pi = (\Pi^H, \Pi^L) \in \Re^{2n}$. The game of interest belongs to:

$$\Omega \equiv \Big\{ \Pi \in \Re^{2n} \quad | \quad \pi_{k+1}^\zeta \geq \pi_k^\zeta, \forall\, \zeta, \forall\, k \text{ with strict inequality for some } k; \\ \pi_n^H > \pi_1^L, \pi_n^L > \pi_1^H;\ \pi_n^H \geq \pi_n^L \Big\}. \tag{3.1}$$

The first set of inequalities in Eq. (3.1) imply that a player taking a particular action is no worse off when the number of opponents taking the same action increases. The next two inequalities require that everyone playing a common action is a strict Nash equilibrium. The last inequality means that the equilibrium where everyone plays H, denoted **H**, is better than where everyone plays L, denoted **L**. Figure 3.2 depicts an example of a three-person coordination game with H is stag and L is hare. Now the following preliminary result is straightforward:

Lemma 2 *If* $\Pi \in \Omega$ *then the only pure strategy equilibria of* $G(n, \Pi)$ *are two strict Nash, viz.* **H** *and* **L**.

Proof It suffices to show that none of $k = 1, 2, \ldots, n-1$ satisfies both $\pi_{n-k}^L > \pi_{k+1}^H$ and $\pi_k^H > \pi_{n-k+1}^L$, since otherwise the pure strategy profile of k players choosing H and $(n-k)$ players choosing L would be Nash. Adding the above two inequalities yields

$$-(\pi_{n-k+1}^L - \pi_{n-k}^L) > \pi_{k+1}^H - \pi_k^H$$

which contradicts the definition of the Ω set. ∎

Any of the Nash refinements, including the strategic stability of Kohlberg and Mertens, is powerless in selecting between these two strict Nash equilibria. Pareto efficiency is compatible with equilibrium play, so neither an incentive problem nor conflict exists. However, it is not clear whether players will be able to reach this outcome in a noncooperative situation where no direct communication is allowed. In short, strategic uncertainty matters.

3.4 Adjustment Dynamics

3.4.1 The Model

Time is continuous from $t = 0$ to ∞. The game $G(n, \Pi)$ is played repeatedly in a society with a continuum of identical players.[6] At every point in time, each player is matched to form a group with the other $(n-1)$ players, who are randomly drawn from the population playing the game anonymously. All players behave rationally, choosing a strategy to maximize expected discounted payoffs. Because of anonymity, they engage in this maximization without taking into account strategic considerations such as reputation, punishment, and forward induction.

The key assumption is that not every player can switch actions at will. Every player needs to make a commitment to a particular action in the short run. More specifically, we assume that the opportunity to switch actions arrives randomly, following a Poisson process with parameter λ, the mean arrival rate. It is further assumed that this process is independent across the players and that there is no aggregate uncertainty. The strategy distribution in the society as of time t can be thus described as y_t, the fraction of the players that are committed to action H as of t. Due to the restriction mentioned above, the social behavior pattern y_t changes continuously over time with its rate of change belonging to $[-\lambda y_t, \lambda(1-y_t)]$. Furthermore, any feasible path necessarily satisfies $y_0 e^{-\lambda t} \leq y_t \leq 1-(1-y_0)e^{-\lambda t}$, where the initial condition y_0 is given exogenously or "by history."

[6]Boylan [1992] verifies that, if the population is countably infinite, there exists a probability space and a sequence of random variables which correspond to a random matching process such that the law of large numbers can nicely apply, i.e. there is no aggregate uncertainty. Green [1989] offers some big enough probability space to encompass the continuum model.

When the opportunity to switch arrives, players choose the action which results in the higher expected discounted payoffs, recognizing the future path of y as well as their own inability to switch actions continuously. The value of playing action H instead of L as of time t is equal to

$$\begin{aligned}\Phi(y_t) &= \sum_{k=1}^{n} \binom{n-1}{k-1} y_t^{k-1}(1-y_t)^{n-k}\pi_k^H - \sum_{k=1}^{n} \binom{n-1}{k-1} y_t^{n-k}(1-y_t)^{k-1}\pi_k^L \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} y_t^{k-1}(1-y_t)^{n-k}\phi_k, \end{aligned} \tag{3.2}$$

where $\phi_k \equiv \pi_k^H - \pi_{n-k+1}^L$ is nondecreasing in k. Given the opportunity, players commit to take H if $V_t > 0$ and to L if $V_t < 0$ and are indifferent if $V_t = 0$, where

$$V_t \equiv (\lambda + r) \int_0^{\infty} \Phi(y_{t+s}) e^{-(\lambda+r)s} ds \tag{3.3}$$

with $r > 0$ being the discount rate. We define $\rho \equiv \frac{r}{\lambda}$ to be the effective discount rate or the degree of friction. Therefore, $\{y_t\}_{t=0}^{\infty}$ is an equilibrium path from y_0 if its righthand derivative exists and satisfies

$$\dot{y}_t^+ = \begin{cases} \lambda(1-y_t) & \text{if } V_t \geq 0, \\ -\lambda y_t & \text{if } V_t \leq 0, \end{cases} \tag{3.4}$$

for any t. This states that all players currently playing action H (respectively L), if given the chance, switch to L (resp. H), when $V_t <$ (resp. $>$)0.

3.4.2 Characterization

We borrow from Matsui and Matsuyama [1991] the following terminology. A state y is called *accessible* from y', if an equilibrium path from y' that reaches or converges to y exists. It is called *globally attractive* if it is accessible from any $y' \in [0, 1]$. A state y is called *absorbing*[7] if a neighborhood U of y exists such that any equilibrium path from U converges to y. It is *fragile* if it is not absorbing. The

[7] Although this is the same concept as *asymptotically stable* according to standard terminology in dynamical systems, we simply use *absorbing* due to the presence of multiple paths. It should be emphasized that this is nothing to do with the Markov processes.

definition does not rule out the possibility that a state may be both fragile and globally attractive, or that a state may be uniquely absorbing but not globally attractive. However, we will show that these situations will not occur in this model.

We will show that the parameter Π characterizes the game to be in one of three sets Ω_0, Ω_1 and Ω_{01}, where the state $y = 0$ is globally attractive in Ω_0, the state 1 is globally attractive in Ω_1, and both states are absorbing in Ω_{01}. For this purpose, we need the coefficients

$$\alpha_k(n,\rho) \equiv \frac{1+\rho}{n}\prod_{j=k}^{n}(\frac{j}{j+\rho}) \text{ and } \beta_k(n,\rho) \equiv \alpha_{n-k+1}(n,\rho). \tag{3.5}$$

For notational simplicity, we suppress (n, ρ) whenever possible. We denote the vectors $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. The reader might be embarrassed with the complicated forms of the coefficients α_k's and β_k's. According to the lemma below, however, they play a role as weights, putting higher (resp. lower) weight on larger k in α (resp. β). The weight is spread equally over all the k's as the friction disappears, while it concentrates on an extreme k as the friction grows without bound.

Lemma 3 *For any n given, (a)* $\sum_{k=1}^{n}\alpha_k = \sum_{k=1}^{n}\beta_k = 1,\ \forall\rho$;
(b) $\alpha_{k+1} > \alpha_k$ *and* $\beta_{k+1} < \beta_k,\ \forall k, \rho \in (0,\infty)$; *(c)* $\lim_{\rho\to 0}\alpha_k = \lim_{\rho\to 0}\beta_k = \frac{1}{n},\ \forall k$;
(d) $\lim_{\rho\to\infty}\alpha = (0, ..., 0, 1)$ *and* $\lim_{\rho\to\infty}\beta = (1, 0, ..., 0)$.

Proof is deferred to the Appendix. The " $\cdot$ " denotes the inner product of two vectors. For example, $\alpha \cdot \Pi^\zeta = \sum_{k=1}^{n}\alpha_k\pi_k^\zeta$, etc. We derive proposition 3 together with the definition of the sets:

$$\Omega_0 = \{\Pi \in \Omega | \alpha \cdot \Pi^H \leq \beta \cdot \Pi^L\}, \tag{3.6}$$

$$\Omega_1 = \{\Pi \in \Omega | \beta \cdot \Pi^H \leq \alpha \cdot \Pi^L\}, \tag{3.7}$$

$$\Omega_{01} = \Omega \backslash (\Omega_0 \bigcup \Omega_1). \tag{3.8}$$

Proposition 3 *The state y is globally attractive iff $\Pi \in \Omega_y$ for either $y = 0$ or $y = 1$; both $y = 1$ and $y = 0$ are absorbing iff $\Pi \in \Omega_{01}$. Moreover, if an absorbing state, y, is globally attractive, then it is a unique absorbing state in $[0, 1]$ and any other state must be fragile.*

Proof Provided in Appendix. ∎

Proposition 4 *(a) In the limit as $\rho \to 0$, the state $y = 1$ (resp. $y = 0$) is uniquely absorbing and globally attractive iff $\frac{1}{n}\sum_{k=1}^{n} \pi_k^H >$ (resp. $<$) $\frac{1}{n}\sum_{k=1}^{n} \pi_k^L$; (b) in the limit as $\rho \to \infty$, both states are absorbing and no state globally attractive.*

Proof Part (a) is clear from Lemmas 3(b) and (c). As ρ goes to infinity, Lemma 3(d) together with Eq. (3.1) implies that both Ω_0 and Ω_1 become empty, while Ω_{01} eventually occupies the whole set Ω. ∎

Keep in mind that the smaller (larger) size of ρ implies the more (less) patience and/or a shorter (longer) duration of an action commitment.[8] The smaller the degree of friction ρ gets, the more the long run equilibrium tends to rely on the payoff matrix specification and the less on the initial position of strategic uncertainty, and vice versa. As players are more patient and/or it costs less to switch their choices, the steady state will be the good Pareto efficient equilibrium

[8]Indeed, $r \to 0$ implies that players are more concerned about the future. That $\lambda \to \infty$ might have two opposite effects: players are less concerned about the future whilst the current strategy distribution becomes less important. Nevertheless, a strictly positive r guarantees that the second effect always dominates the first one. Therefore, the smaller ρ gets, the more players worry about the future.

as long as the "static" unweighted average from H exceeds that from taking L. The interpretation is as follows: Suppose you are going to play a one shot game $G(n, \Pi)$. Since you are confronted with $(n-1)$ opponents, there are n possible events, denoted $A_{k-1},\ k = 1, 2, ..., n$, where exactly $(k-1)$ opponents choose action H. You assume that each of those events takes place with equal probability $\frac{1}{n}$, and makes your best response. Notice that this necessarily implies some correlation among your opponents' choices.

On the other extreme case of ρ approaching to infinity, sometimes called best response dynamics, both states may obtain in the long run and exactly which one would come out depends solely upon what the initial state was. In fact, Matsui [1992] verifies an equivalence between the best response dynamics and a static equilibrium concept attributed to Swinkels [1992a]. This notion, called an evolutionary stability with equilibrium entrants, imposes an additional restriction on the qualification of mutants, thus is weaker than the traditional evolutionary stability. Notice that the connection[9] of "myopic" replicator dynamics to strategic stability or rationalizability would be vacuous in coordination games, because both Nash equilibria simply survive the strict iterated admissibility. Such indeterminacy is resolved if we perturb the dynamical system with a constant flow of mutations and experimentations. Kim [1992b] not only characterizes the long run ergodic equilibrium of the resulting stochastic dynamics in the limit as the probability of mutations approaches to zero, but also provides clear and intuitive comparions between the equilibrium selections derived. The reader is urged to refer to that paper for details.

[9]Refer to Swinkels [1992b] and references therein.

3.5 Equivalence with Global Perturbation

The global perturbation approach of Carlsson and Van Damme [1990, 1991] is based on the idea that players are uncertain not only about the payoffs but also their modeling of the game itself. Each player i will receive a private signal θ_i that provides an unbiased estimate of θ, but the signals are noisy so the true value of θ is not common knowledge. Let Θ be a random variable and let $\{E_i\}_{i=1}^n$ be an n tuple of i.i.d. random variables, each having zero mean. The E_i are independent of Θ, with a continuous density and a support within $[-1, 1]$. For $\varepsilon > 0$, write

$$\Theta_i^\varepsilon = \Theta + \varepsilon E_i.$$

Notice that ε measures perfectness of the common knowledge.

Given this structure, we formally define the incomplete information game $G^\varepsilon(n, \Pi)$ described by the following rules: A realization $(\theta, \theta_1, ..., \theta_n)$ of $(\Theta, \Theta_1^\varepsilon, ..., \Theta_n^\varepsilon)$ is drawn, player i is informed only about θ_i and chooses between H and L, each player i receives payoffs as determined by $G(n, \Pi(\theta))$ and the action taken. Even if player i knows upon having observed θ_i that the true θ lies in $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and that all other θ_j's in $[\theta_i - 2\varepsilon, \theta_i + 2\varepsilon]$, this fact is not common knowledge. Now suppose that θ_j is realized as, say, $\theta_i - 2\varepsilon$, then player j knows that θ lies in $[\theta_j - \varepsilon, \theta_j + \varepsilon]$, thus in $[\theta_i - 3\varepsilon, \theta_i + \varepsilon]$, and that θ_i must be in $[\theta_j - 2\varepsilon, \theta_j + 2\varepsilon]$, thus in $[\theta_i - 4\varepsilon, \theta_i]$. The problem is that player i does not know that player j knows that θ_i lies in $[\theta_i - 4\varepsilon, \theta_i]$. This argument applies also to all the other less extreme realizations of θ_j. Lack of common knowledge expands all the way down, and thus enables remote areas of dominated strategies $(-\infty, \underline{\theta})$ and $(\bar{\theta}, \infty)$ to exert an influence, however tiny ε might be as far as it is strictly positive.[10]

We confine our attention to the perturbation p_k^H (resp. p_{n-k}^L) that satisfies the

[10] Such remote areas play an important role in Rubinstein [1989] as well.

following two conditions:

Assumption 1 *(a) They are continuous, monotonically increasing (resp. decreasing) in* θ*, and unbounded above and below,* $\forall k$*; (b) the original unperturbed game obtains with* $\theta = 0$*, i.e.* $p_{\cdot}^{\zeta}(0) = \pi_{\cdot}^{\zeta}$ *for* ζ *=H,L.*

Let $\bar{\theta}$ (resp. $\underline{\theta}$) the infimum (resp. supremum) of θ's such that H (resp. L) is a strictly dominant strategy in a game with payoff realization θ. By assumption 1 above, it is obvious that $-\infty < \underline{\theta} < 0 < \bar{\theta} < +\infty$.

Assumption 2 *The* Θ *is uniformly distributed over an interval* $\supset [\underline{\theta}, \bar{\theta}]$.

One might ask whether this is a big perturbation. Answer is both no and yes. As it turns out, our goal is to see what happens at the exactly original games in the limit as the common knowledge about payoffs is almost perfect, after characterizing the equilibrium behavior assuming this perturbation. Neverthless, players are taking very different situations from the original game into account through his contemplation process. Uniformity part would play an important role, since only order but not location of the realizations of random noise variable matters. We believe that our main points would still emerge without this restriction, but we have not verified that this is the case. A guess on the relaxation of this assumption will be made in the last section.

Under these assumptions, an iterative elimination of strictly dominated strategies, namely *strict iterated admissibility,* will be applied. The next lemma shows that the Bayesian Nash equilibrium has the cutoff property, and that the game considered here is indeed dominance solvable.

Lemma 4 *If Assumption 1 and 2 hold, then the equilibrium is characterized by cutoff* θ_{GP} *such that player i optimally chooses H (resp. L) iff* $\theta_i >$ *(resp.* $<$*)*θ_{GP}*. Furthermore,* θ_{GP} *is a unique root of the equation* $\frac{1}{n}\sum_k p_k^H(\theta) = \frac{1}{n}\sum_k p_k^L(\theta)$.

Proof Provided in Appendix. ▮

Recall that the perturbed game will correspond to the original unperturbed game when $\theta = 0$. We are interested in what happens at $\theta = 0$ in the limit as the common knowledge about payoffs becomes arbitrarily perfect, i.e. ε goes to zero. Recall that $|\theta_i| < \varepsilon$ if $\theta = 0$. So if $\theta_{GP} >$ (resp. $<$)0 for ε small enough then $\theta_i <$ (resp. $>$)θ_{GP} for all i when $\theta = 0$. In other words, if $\theta_{GP} <$ (resp. $>$)0, then when $\theta = 0$ and ε is sufficiently small, the results of iterative strict dominance prescribe that all players play H (resp. L). So we say that the equilibrium **H** in the unperturbed game is *robust with respect to global perturbation* if $\theta_{GP} < 0$, and that **L** is robust if $\theta_{GP} > 0$. Recall that the state y be the fraction of population taking action H. Then argument thus far yields:

Main Theorem *The* $y = 1$ *(resp.* 0*) is the unique absorbing and globally attractive state in the limit as* $\rho \to 0$ if and only if *action H (resp. L) is robust with respect to global perturbation.*

A couple of papers in the literature deserve some mention. Harsanyi [1973] uses a similar perturbation to justify mixed strategy equilibria. His formulation requires, however, that the value of θ be common knowledge so observing θ_i implies knowing the realization of E_i, but not E_{-i}'s, and that the payoff of player i depends on θ_i rather than on θ. Fudenberg, Kreps, and Levine [1988] argues that an equilibrium that is unreasonable (in the sense of being eliminated by Nash refinements) in a given game may not be unreasonable in nearby games. They assert that every strict equilibrium is reasonable and they roughly show that every normal form perfect equilibrium can be approximated by strict equilibria of nearby games, hence, that any such equilibrium is reasonable as well. Their paper differs from global perturbation in the definition of nearness of games and in the assumption that only the analyst does not know the payoffs, the payoffs are, however, common knowledge among the players themselves.

We now calculate selection on the basis of Harsanyi and Selten's [1988] risk dominance, and refute its equivalence to global perturbation, thus to the limit adjustment dynamic outcome. Refer to the tracing procedure discussed in Section 3.2. From (i) and (ii) of the preliminary theory, the outside observer concludes that player i takes H according to $\pi_n^H z_i + \pi_1^H(1-z_i) > \pi_1^L z_i + \pi_n^L(1-z_i)$, or

$$z_i > \mu \equiv \frac{\pi_n^L - \pi_1^L}{(\pi_n^H - \pi_1^H) + (\pi_n^L - \pi_1^L)}.$$

Using step (iii), the outside observer forecasts player i's strategy as $q_i = (1-\mu)[\mathrm{H}] + \mu[\mathrm{L}]$, with different q_i being independent. The tracing procedure to find a distinguished path in the graph of the correspondence from a linear combination of the naive $G(q)$ and $G(n,\Pi)$ to the set of Nash equilibria is simple in the case at hand. Player i's expected payoff difference associated with H and L in $G(n,\Pi)$ when each of the opponents follows the strategy q_{-i} will be

$$\begin{aligned} & \sum_{k=1}^{n} \binom{n-1}{k-1} (1-\mu)^{k-1}\mu^{n-k}\pi_k^H - \sum_{k=1}^{n} \binom{n-1}{k-1} \mu^{k-1}(1-\mu)^{n-k}\pi_k^L \\ \equiv \ & \Phi(1-\mu). \end{aligned}$$

Recalling that $\Phi(0) < 0 < \Phi(1)$ and Φ is monotonic increasing, write μ_{RD} the unique root of the equation $\Phi(1-\mu) = 0$. Hence, each player's best response against q would be H (resp. L) iff $\mu <$ (resp. $>$)μ_{RD}. Now it is not difficult to verify the non-equivalence part, since the payoff Π satisfying the condition $\frac{1}{n}\sum_k \pi_k^H = \frac{1}{n}\sum_k \pi_k^L$ does not generically satisfy the risk dominance solution $\Phi(1-\mu) = 0$, for $n \geq 3$. In this course, one notes that they just happen to be equal when $n = 2$.

3.6 Strong Result in Two Person Games

We have assumed that the speed of adjustment, represented by Poisson arrival parameter λ, are identical over the whole population members. This seems not a severe restriction since we have studied symmetric games. Neverthless, we can in principle incorporate asymmetric speed of adjustment into our n-person games by assuming that each population i has a Poisson arrival rate $\lambda_i,\ i = 1, 2, ..., n$. A fair amount of numerical simulations indicate that our main thorem does depend on these numbers. However, a very strong result holds in any coordination game with $n = 2$. We show that, in any – symmetric or not – 2×2 games, the limit dynamic equilibrium outcome is equivalent to global perturbation (and to risk dominance), despite asymmetric speeds of adjustment as long as the proportion of adjustment speeds is fixed.

A stage 2×2 coordination game, depicted in Figure 3.3, is played repeatedly by two continuua of identical players. At every point in time, each player from population 1 is randomly matched to the other player from population 2. We assume that all members of population i has the adjustment speed λ_i, where $i = 1, 2$. Denote y_t^i to be the ith population fraction who is currently choosing action H as of time t. For notational simplicity, let $\mathbf{1} \equiv (1, 1)$ and $\mathbf{0} \equiv (0, 0)$. The value of playing action H instead of L as of time t is equal to

$$\Phi^i(y_t^1, y_t^2) = y_t^j - \mu_i,\ i = 1, 2,\ j \neq i,$$

where $\mu_i \equiv \frac{d_i - b_i}{a_i - c_i + d_i - b_i}$. Note that Φ^i only depends on y^j and $\frac{\partial \Phi_i}{\partial y_j} = 1 > 0$. Given the opportunity to move, each player from population i chooses his action maximizing

$$V_t^i \equiv (\lambda_i + r) \int_0^\infty \Phi^i(y_{t+s}^1, y_{t+s}^2) e^{-(\lambda_i + r)s} ds.$$

We have[11]

[11]The main thorem of Fudenberg and Harris [1992] states the same result using a completely

Proposition 5 *Fix any* 2×2 *coordination game. In the limit as* $\rho_i \to 0$ *for* $i = 1, 2$, *with* $\frac{\rho_1}{\rho_2} = \delta \in (0, 1]$ *fixed, the uniquely absorbing and globally attractive state is* $\mathbf{y} = \mathbf{1}$ *(resp.* $\mathbf{y} = \mathbf{0}$*) iff* $\mathbf{H}$ *(resp.* $\mathbf{L}$*) is robust with respect to global perturbation.*

For notational simplicity, let $\rho_2 = \rho$. It is easy to check that

$$\frac{\lambda_1 + r}{\lambda_1 + \lambda_2 + r} = \frac{1 + \delta\rho}{1 + \delta + \delta\rho} \text{ and } \frac{\lambda_2 + r}{\lambda_1 + \lambda_2 + r} = \frac{\delta + \delta\rho}{1 + \delta + \delta\rho}.$$

We state Lemma 5 with the modified definition of the sets:

$$\Omega_{\mathbf{0}}(\delta, \rho) \equiv \{(\mu_1, \mu_2) \in (0, 1)^2 | \mu_2 \geq F_{\delta,\rho}(\mu_1)\}, \tag{3.9}$$

$$\Omega_{\mathbf{1}}(\delta, \rho) \equiv \{(\mu_1, \mu_2) \in (0, 1)^2 | \mu_2 \leq 1 - F_{\delta,\rho}(1 - \mu_1)\}, \tag{3.10}$$

$$\Omega_{\mathbf{01}}(\delta, \rho) \equiv (0, 1)^2 \backslash (\Omega_{\mathbf{0}} \bigcup \Omega_{\mathbf{1}}), \tag{3.11}$$

where

$$F_{\delta,\rho}(\mu) = \begin{cases} f_{\delta,\rho}(\mu) & \text{for } 0 < \mu \leq \frac{1+\delta\rho}{1+\delta+\delta\rho} \\ f_{\delta,\rho}^{-1}(\mu) & \text{for } \frac{1+\delta\rho}{1+\delta+\delta\rho} \leq \mu < 1 \end{cases}$$

and

$$f_{\delta,\rho}(\mu) \equiv 1 - \frac{(1 + \delta + \delta\rho)^{\rho}}{(1 + \delta\rho)^{1+\rho}} \mu^{1+\rho}.$$

Lemma 5 *The state* $\mathbf{y}$ *is uniquely absorbing and globally attractive iff* $(\mu_1, \mu_2) \in \Omega_{\mathbf{y}}$ *for either* $\mathbf{y} = \mathbf{0}$ *or* $\mathbf{1}$; *both* $\mathbf{1}$ *and* $\mathbf{0}$ *are absorbing iff* $(\mu_1, \mu_2) \in \Omega_{\mathbf{01}}$.

The Lemma above is verified in Appendix. In this course, one can be aware that the history dependency region $\Omega_{\mathbf{01}}$ is the smallest when $\delta = 1$ (ie, $\rho_1 = \rho_2 = \rho$),

different evolutionary dynamics with aggregate uncertainty, but the class of games they study is a narrower symmetric games.

and that it expands as δ departs away from 1 above or below. This implies that the discrepancy of frictions between populations causes as severe indeterminacy problem as the absolute size of frictions themselves. Finally, Proposition 4 obtains simply by letting $\rho \to 0$ with δ fixed and Carlsson and Van Damme [1990] result that the equilibrium selection based on global perturbation is equivalent to risk dominance in any 2×2 game.

3.7 Applications

3.7.1 Pure Coordination

Consider a two person pure coordination game. It is often argued that, even without preplay communication, introspection alone will lead players to coordinate on the Pareto optimum. This intuition is confirmed as reasonable even in broader definition of pure coordination games. A pure coordination game specifies the payoff parameters to be

$$\pi_k^H \text{ (resp.} \pi_k^L) = \begin{cases} a \text{ (resp.} b) & \text{for } \kappa \le k \le n \\ c & \text{otherwise} \end{cases}$$

where κ may be any of $2, 3, ..., n$, and $a > b > c$.

Corollary 2 *There exists $\bar{\rho} > 0$ such that the only uniquely absorbing and globally attractive state is $y = 1$ for any n, y_0, and $\rho \in (0, \bar{\rho})$. Equivalently, the only equilibrium selected based on the global perturbation must be the Pareto efficient* **H** *for any n.*

Proof Since $\frac{1}{n}\sum_k \pi_k^H = \frac{n-\kappa+1}{n}a > \frac{n-\kappa+1}{n}b = \frac{1}{n}\sum_k \pi_k^L$ always holds, it is straightforward that, as $\rho \to 0$, the Ω_1 set will ultimately occupy the whole Ω and the remaining region Ω_0 and Ω_{01} be empty sets. The second part is direct from our main theorem. ∎

3.7.2 Stag Hunt

The most general payoff specification that includes the game discussed in the expository section is as follows:

$$\pi_k^L = x \in (0,1) \text{ all } k$$

$$0 = \pi_1^H \leq \pi_2^H \leq ... \leq \pi_n^H = 1.$$

Besides its practical applicability, this game has a couple of merits to analyze. First, the Pareto optimality is at odds with the security, so which outcome would actually appear may be controversial. Second, it reduces the Ω sets to a one dimensional space, which makes the results extremely intuitive and facilitates numerical studies. Recalling that $\cdot$ denotes a dot product of two vectors, we define

$$u(n,\rho) \equiv \alpha \cdot \Pi^H \text{ and } \ell(n,\rho) \equiv \beta \cdot \Pi^H \tag{3.12}$$

where α_k's and β_k's are as in Eq. (3.5). Directly applying proposition 1 and 2 yields:

Lemma 6 *(a) The state $y = 1$ is globally attractive iff $x \geq u(n,\rho)$; $y = 0$ is globally attractive iff $x \leq \ell(n,\rho)$; both $y = 1$ and $y = 0$ are absorbing iff $\ell(n,\rho) \leq x \leq u(n,\rho)$;*
(b) in the limit as $\rho \to 0$, the state $y = 1$ (resp. $y = 0$) is globally attractive iff $x <$ (resp. $>$)$\frac{1}{n}\sum_k \pi_k^H \equiv x_{LD}$; (c) in the limit as $\rho \to \infty$, both states are absorbing.

Corollary 3

$$x_{LD} = \frac{1}{n}\sum_{k=1}^{n} \pi_k^H = x_{GP} \neq x_{RD}$$

The LD in the subscript stands for 'limit dynamic'. Corollary 3 immediately follows from Lemma 6 and Carlsson and Van Damme [1991].

3.8 Experimental Implications

A brief survey of Van Huyck et al. [1990, 1991] experimental results is offered. Each treatment typically lasts ten stages but the number of stages was not announced in advance in some experiments. A summary statistic of subjects' strategy choices was publicly announced after each stage. At the end of each experiment, subjects were paid the sum of their payoffs in the games they played. In each of the games, each player i chooses a pure strategy, denoted e_i and called effort, from the set $\{1, ..., 7\}$. In each stage, each player's payoff was determined by his own effort and a simple summary statistics of those of the players in his group. This statistic was either minimum or median of group effort choices. The parameter values were given for these normal forms[12] to be of coordination games with seven strict Pareto ranked symmetric pure strategy Nash equilibria. In every game, the payoff dominance selects all players' choosing the highest effort, i.e. **7**, irrespective of the number of subjects in a group. With respect to group size, a large group consists of 14 to 16 players whilst a small group of only two persons.

Despite payoff dominance, in large group minimum treatments subjects initially choose widely dispersed efforts and then rapidly approached the lowest effort, $e = 1$: up to 84% of the subjects reached that effort within a few stages. In one treatment in which the parameters were adjusted so as for the highest effort $e = 7$ to be weakly dominant, approximately 96 percentage reached that effort by the fifth stage. This result may justify our maintained assumption that no strictly dominated strategy will be played at all.[13] In small group experiments, subjects' initial choices varied substantially and then drifted over time with no clearly

[12] Van Huyck et al.1990 article for minimum treatments and 1991 research for median ones contain parameter values actually used in the experiments and the resulting normal forms.

[13] No conflict arises with Cooper et al. [1990] experimental evidence, which just asserts that any addition or deletion of dominated strategies may affect the equilibrium actually selected.

discernable trend. By contrast, subjects in every median treatment converged completely and promptly to the Nash equilibrium determined by the "historical accident" of their initial stage median, despite considerable variation in the initial median across treatments. In brevity, it exhibits a strong history dependency. Finally, in large group median experiments with pure coordination game, players move swiftly to the Pareto best equilibrium action. This last observation can be at least partially explained by our corollary 2 and the fact that subjets were allowed to switch their choices every period.

Our simple model captures many salient features that were reported above. To see this, we consider a stag hunt game as follows:

$$\pi_k^H = \begin{cases} 1 & \text{if } \kappa \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

where κ denotes the minimum number of players necessary for a successful stag hunt. Note that the minimum rule specifies $\kappa = n$ and that the median vote does $\kappa = \frac{n+1}{2}$. Plugging into Eq. (3.12) gives rise to

$$u = \sum_{k=\kappa}^{n} \alpha_k \quad \text{and} \quad \ell = \sum_{k=\kappa}^{n} \beta_k,$$

with α_k and β_k as defined in Eq. (3.5). Remember from lemma 6 that the steady state could be **H** and **L** regardless of the initial state, respectively, according as $x < \ell$ and $x > u$. Certainly there might exist an equilibrium path converging to, say, **H** when $x > u$, if the initial population fraction of stag hunters is very high. However, we execute the numerical analysis as if the globally attractive state was globally stable. This is silly but can be tolerated reflecting the fact that the two regions of global attraction roughly offset each other. In the case where $x \in [\ell, u]$, exactly which equilibrium will be obtained in the long run hinges on y_0, the historical accident of initial states. For the sake of calculation, we impose the monotonicity requirement, that is, only the paths monotonically converging to either **H** and **L** will be taken into account. That is, any cyclical path is ruled out. Deterministic nature of the present dynamic together with monotonicity imply the existence of a unique critical value of x, below which the path converges to **H**, and vice-versa.

We assume throughout that x is uniformly distributed over $[0, 1]$. Two remarks are in order. First, the strategic uncertainty as has been understood should imply the distribution over the initial y_0 with x being fixed. It causes numerically little problem to consider y_0 as deterministic and instead x as uncertain. Another justification might be the uncertainty on the part of the experimenter about subjects' subjective evaluations of the fixed monetary compensation x. Second, relaxing reasonably the uniformity of the x-distribution here only seems to make our result stronger. As in Van Huyck et al.'s experiments, we let $n = 2$ for a small group and $n = 15$ for a large group.

Let $y(x)$ denote the inverse function of

$$\sum_{k=\kappa}^{n} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} y^{k-1}(1-y)^{n-k} = x,$$

where the left (resp. right) hand side is the expected payoff from H (resp. L). Deterministic nature of the model and the monotonicity of paths make it clear that a player should choose action H (resp. L) if $y_0 >$ (resp. $<$)$y(x)$ in the intermediate history dependency region, given an opportunity to switch. The probability that the steady state be the Pareto inferior Nash **L** will be at least approximately:

$$\begin{aligned} \Pr(\mathbf{L} \text{ is the steady state}) &= \Pr(x \geq u) + \Pr(\ell < x < u, y_0 < y(x)) \\ &= (1-u) + \int_{\ell}^{u} y(x) dx, \end{aligned}$$

where u and ℓ are calculated using Eq. (3.5).

In small group treatment, the probability that the steady state is **L** would be $(1 - \frac{1+\rho}{2+\rho}) + \int_{\ell}^{u} x dx = 0.5$ regardless of ρ. Under a large group minimum rule, this probability will be expressed as $(1-u)+\int_{\ell}^{u} x^{\frac{1}{14}} dx$. Table 3.2 provides several simulations according to varying parameters.[14] The range of x in which the Pareto inferior equilibrium **L** could be selected irrespective of the initial states is very broad, unless subjects are extremely impatient. On the other hand, with a big ρ value, the portion of which both strict equilibria are absorbing is large. But even in such a situation, the basin of attraction with respect to initial strategic uncertainty is much larger for **L** than that for **H** under maintained assumption of the path monotonicty. These are reflected on the fact that the steady state is likely to settle down on the inferior equilibrium **L** with probability of at least 93.3 percent and up to 97 percent. The high probability of attaining the Pareto inferior equilibrium is consistent with coordination failures which were observed in Van Huyck et al's experimental results.

Table 3.3 analogously analyzes the large group median treatments. The probability that the Pareto inferior Nash equilibrium **L** will be selected as the long run state is shown to be stable around 46 percent. The point here is that, for each ρ given, a relatively wide range between ℓ and u indicates a strong dependence on the initial state, or put differently, "historical accident." For instance with $\rho = 1$, the history dependence region $[.008, .125]$ of a minimum rule is in sharp contrast to $[.300, .767]$ of median vote. Comparison of tables show that this characteristic is fairly robust to the somewhat arbitrary parameter ρ size.

We close the section with some loose comments on Harrison and Hirshleifer's [1989] public goods provision experiments. Consider two popular models, the "weakest link" and the "best shot." The weakest link model refers to the situation where failure of even a single person brings about miserable ruins to all, for example military units defending segments of the front against an enemy offen-

[14] All simulations were carried out using *Mathematica.*

sive. In contrast, the best shot refers to the case where only one's provision or success is enough for all, such as rats trying to bell the cat. HH convincingly argues that the "free-rider" problem would be less (more) serious, thus cooperation would be more (less) likely to obtain, in the weakest link (best shot, respectively) model. Reflecting the fact that the weakest link is strategically equivalent to the stag hunt game under the minimum rule, their insight and the basic theme here seemingly contradict each other. This is not the case. Take the example of military units defending segments of the front against an enemy offensive. If all other units are successfully defending their own segments and if this fact is common knowledge then it certainly would be in my interest to defend my own. However, once even a single segment is broken through, running away will be everyone else's best response. How does one know the others are doing well? As an obvious guess, it seems likely that some means of signalling, such as cheap talk and sequential move structure, could enhance the possibility of cooperation. On the contrary, the actual failure of or little doubt about the perfect defense will make the good equilibrium collapse.[15] We view this as an underlying reason for HH's experimental outcomes, in which subjects show a substantial cooperation with the sequential protocol while little clearcut evidence on cooperation or behavioral pattern is perceived with the sealed bid protocol.

3.9 Concluding Remarks

The present paper has of course some shortcomings, especially in its critical dependence on a somewhat arbitrary parameter ρ, the effective discount rate or friction. Uniformity of the distribution of random noise variables looks to be also restrictive, although a scrutiny of the proofs in Carlsson and Van Damme [1990]

[15] It is a *contagious* equilibrium in Kandori's [1992] sense.

might suggest a relaxation of this assumption. Our conjecture is as follows: under a general distribution with compact support and given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ satisfying $\lim_{\varepsilon \to 0} \eta = 0$ such that player i optimally chooses action H (resp. L) if his private signal $\theta_i > \theta_{GP} + \eta$ (resp. $< \theta_{GP} - \eta$). Moreover, this almost dominance solvability in the limit is reduced to exact dominance solvability under uniform distribution.

Effort is needed to generalize in encompassing multi actions and/or asymmetric payoffs. Technical difficuties arise from the large amount of case distinctions and calculations. With m actions, we have to consider $2^m - 1$ number of $\Omega_{(\cdot)}$ sets, where only $\boldsymbol{a}$ is globally attractive if $\Pi \in \Omega_a$ and only $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, ..., $\boldsymbol{a_i}$ are absorbing if $\Pi \in \Omega_{a_1 a_2 \ldots a_i}$ for $2 \leq i \leq m$. Payoff asymmety in n person m action games require considering an n^m dimensional space. While there is, at least in principle, no reason why adjustment dynamics or global perturbation fails to be well-defined in the general setting, it is known that risk dominance may well be troublesome because of intransitivities between strict equilibria. In view of our corollary 2, this line of research seems to include as a special example the former part of Kandori and Rob [1992], which abandons risk dominance even in a two person m-action coordination game. Experimental results along the lines of Harrison and Hirshleifer [1989], Cooper et al. [1992], and Matsui [1991] suggest that it is worth exploring the introduction of a cheap talk argument, thus to see whether and how the possibility of cooperation could be enhanced through a costless preplay communication with more than two players.

Appendix

Lemma 3 For any n given, (a) $\sum_{k=1}^{n}\alpha_k = \sum_{k=1}^{n}\beta_k = 1,\ \forall\rho$;

(b) $\alpha_{k+1} > \alpha_k$ and $\beta_{k+1} < \beta_k,\ \forall k, \rho \in (0,\infty)$; (c) $\lim_{\rho\to 0}\alpha_k = \lim_{\rho\to 0}\beta_k = \frac{1}{n},\ \forall k$;

(d) $\lim_{\rho\to\infty}\alpha = (0,...,0,1)$ and $\lim_{\rho\to\infty}\beta = (1,0,...,0)$.

Proof (a) Via mathematical induction. Checking the case of $n = 2$ is trivial. Supposed that it holds for $n-1$, i.e. $\sum_{k=1}^{n-1}\prod_{j=k}^{n-1}(\frac{j}{j+\rho}) = \frac{n-1}{1+\rho}$, then for n

$$\begin{aligned}\sum_{k=1}^{n}\alpha_k &= \frac{1+\rho}{n}\sum_{k=1}^{n}\prod_{j=k}^{n}(\frac{j}{j+\rho}) \\ &= \frac{1+\rho}{n}[\frac{n}{n+\rho} + \frac{n}{n+\rho}\sum_{k=1}^{n-1}\prod_{j=k}^{n-1}(\frac{j}{j+\rho})] \\ &= \frac{1+\rho}{n}\frac{n}{n+\rho}[1+\frac{n-1}{1+\rho}] = 1.\end{aligned}$$

The fact that $\sum_{k=1}^{n}\beta_k = 1$ is trivial since the elements of the vector β are just a rearrangement of those of α. To check (b),(c) and (d) is straightforward. ∎

Proposition 3 The state y is globally attractive iff $\Pi \in \Omega_y$ for either $y = 0$ or $y = 1$; both $y = 1$ and $y = 0$ are absorbing iff $\Pi \in \Omega_{01}$. Moreover, if an absorbing state, y, is globally attractive, then it is a unique absorbing state in $[0,1]$ and any other state must be fragile.

Proof First of all, notice that $\Phi(0) = \pi_1^H - \pi_n^L < 0 < \Phi(1) = \pi_n^H - \pi_1^L$ and that Φ is strictly increasing, since

$$\Phi'(y) = (n-1)\sum_{k=0}^{n-2}\begin{pmatrix} n-2 \\ k \end{pmatrix} y^k(1-y)^{n-k-2}[\phi_{k+2} - \phi_{k+1}]\ > 0$$

by the definition of the ϕ function and the nondecreasingness of the π_k sequences.

The outcome **H** can be upset when players have an incentive to deviate for a feasible path from $y = 1$. Because of the monotonicity of Φ, the incentive to deviate is the strongest if all players are anticipated to switch from H to L in the future, i.e. $y_t = e^{-\lambda t}$. Hence, the condition for $y = 1$ being fragile is

$$V_0 = (\lambda + r) \int_0^\infty \Phi(e^{-\lambda s}) e^{-(\lambda+r)s} ds \;\le\; 0,$$

which would be by the change-of-variable technique

$$(1+\rho) \int_0^1 \Phi(y) y^\rho dy \;\le\; 0. \tag{3.13}$$

Using Eq. (3.2), the definition and properties of the Beta and Gamma function,[16] and some algebraic manipulation, Eq. (3.13) becomes

$$\begin{aligned} 0 \;&\ge\; (1+\rho) \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \int_0^1 y^{k+\rho-1} (1-y)^{n-k} dy \\ &=\; (1+\rho) \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \frac{\Gamma(k+\rho)\Gamma(n-k+1)}{\Gamma(n+\rho+1)} \\ &=\; \sum_{k=1}^{n} \alpha_k \phi_k, \end{aligned}$$

or equivalently

$$\sum_{k=1}^{n} \alpha_k \pi_k^H \le \sum_{k=1}^{n} \alpha_k \pi_{n-k+1}^L = \sum_{k=1}^{n} \beta_k \pi_k^L, \tag{3.14}$$

which corresponds to the condition defining the Ω_0 set. We claim: $y = 0$ is globally attractive if and only if $\Pi \in \Omega_0$, and that $y = 1$ is absorbing if and only if $\Pi \in \Omega \backslash \Omega_0$. To prove the if part of $y = 0$ being globally attractive and the only if part of $y = 1$ being absorbing, it suffices to show that, if Eq. (3.14) holds, i.e. $\Pi \in \Omega_0$, a feasible path from $y = 1$ to $y = 0$, $y_t = e^{-\lambda t}$, satisfies the equilibrium condition, i.e. $V_t \le 0 \; \forall t$ along the path. This can be checked as follows:

$$\begin{aligned} V_t \;&=\; (\lambda + r) \int_0^\infty \Phi(y_{t+s}) e^{-(\lambda+r)s} ds \\ &\le\; (\lambda + r) \int_0^\infty \Phi(e^{-\lambda s}) e^{-(\lambda+r)s} ds \;\le\; 0 \; \forall t. \end{aligned}$$

[16] Refer to any text on mathematical statistics.

To prove the if part of $y = 1$ being absorbing and the only if part of $y = 0$ being globally attractive, it suffices to demonstrate that, if $\Pi \in \Omega \backslash \Omega_0$, the equilibrium path is unique and converges to $y = 1$ for y_0 sufficiently close to 1. Reminding that any feasible path from y_0 satisfies $y_t \geq y_0 e^{-\lambda t}$, we get

$$V_0 \geq (\lambda + r) \int_0^\infty \Phi(y_0 e^{-\lambda s}) e^{-(\lambda+r)s} ds.$$

Since the righthand side is strictly positive at $y_0 = 1$ and continuous in y_0, it is still positive for y_0 sufficiently close to 1.

Similarly, the condition for $y = 0$ being fragile combined with the change of variable technique will be

$$\begin{aligned} V_0 &= (\lambda + r) \int_0^\infty \Phi(1 - e^{-\lambda s}) e^{-(\lambda+r)s} ds \\ &= (1 + \rho) \int_0^1 \Phi(y)(1 - y)^\rho dy \geq 0. \end{aligned}$$

Again by the definition of Φ function, the properties of Gamma and Beta function, and some algebraic manipulation, we have

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \phi_k \frac{\Gamma(k)\Gamma(n-k+\rho)}{\Gamma(n+1+\rho)} \\ &= \sum_{k=1}^{n} \beta_k \phi_k, \end{aligned}$$

or equivalently

$$\sum_{k=1}^{n} \beta_k \pi_k^H \leq \sum_{k=1}^{n} \beta_k \pi_{n-k+1}^L = \sum_{k=1}^{n} \alpha_k \pi_k^L,$$

which is the condition defining Ω_1. A symmetric argument as before shows that $y = 1$ is globally attractive if and only if $\Pi \in \Omega_1$, and that $y = 0$ is absorbing if and only if $\Pi \in \Omega \backslash \Omega_1$.

Combining all the facts shown yields the desired result. ∎

Lemma 4 If Assumption 1 and 2 hold, then the equilibrium is characterized by cutoff θ_{GP} such that player i optimally chooses H (resp. L) iff $\theta_i >$ (resp. $<$)θ_{GP}. Furthermore, θ_{GP} is a unique root of the equation $\frac{1}{n}\sum_k p_k^H(\theta) = \frac{1}{n}\sum_k p_k^L(\theta)$.

Proof Notice that the existence and uniqueness of such θ_{GP} are guaranteed by assumption 1(a) and 1(c). As was suggested, we maintain the assumption that no player will choose strictly dominated strategies. Player i will certainly choose H if $\theta_i > \bar{\theta}$: Since the expected value is $E(\Theta|\theta_i^\varepsilon = \theta_i) = \theta_i$, player i knows that H is strictly dominant at each such observation. Consider an observation θ_i of player i slightly below $\bar{\theta}$, such be that $|\bar{\theta} - \theta_i| < 2\varepsilon$. Player i knows that his opponent will play H if $\theta_j > \bar{\theta}$, hence i's payoff if he chooses H at θ_i is approximately

$$\sum_{k=1}^{n} \Pr(\theta_j > \theta_i \text{ for exactly } k-1 \text{ opponents}|\Theta_i^\varepsilon \approx \bar{\theta}) p_k^H(\bar{\theta}) \tag{3.15}$$

$$= \sum_{k=1}^{n} \Pr(E_j > E_i \text{ for exactly } k-1 \text{ opponent}) p_k^H(\bar{\theta}) \tag{3.16}$$

$$= \frac{1}{n}\sum_{k=1}^{n} p_k^H(\bar{\theta}). \tag{3.17}$$

Assumption 2 allows us to conclude that the probability in the Eq. (3.15) is independent of θ_i, at least as long as θ_i lies ε inside the support of Θ. This observation allows us to conclude that this probability must be equal to the a priori probability that E_i is the $k+1$th smallest among the errors. Thus, the Eq. (3.16) ensues, the probability in which is clearly the same for all players. This fact, combined with the assumption that the i.i.d. of E_i has a continuous density, yields Eq. (3.17).

A similar reasoning shows that the expected payoff to action L is at most approximately $\frac{1}{n}\sum_{k=1}^{n} p_k^L(\bar{\theta})$, which is strictly lower than $\frac{1}{n}\sum_{k=1}^{n} p_k^H(\bar{\theta})$ calculated above by the monotonicity assumption 1(a). Hence, if $\theta_{GP} < \bar{\theta}$, there exists $\bar{\theta}^1$ such that H is strictly dominant for any $\theta_i > \bar{\theta}^1$ in the reduced game where player j is constrained to play H when $\theta_j > \bar{\theta}$. In a similar way one can construct $\bar{\theta}^2 < \bar{\theta}^1$

and continuing inductively, we can find sequences $\bar{\theta}^m$ such that H is iteratively dominant for $\theta_i > \bar{\theta}^m$.

On the other hand, starting from the maintained assumption that action L will be chosen when $\theta_i < \underline{\theta}$, we inductively find a sequence $\underline{\theta}^m$ such that L is iteratively dominant for $\theta_i < \underline{\theta}^m$. By the definition of θ_{GP}, it is obvious that $\bar{\theta}^m \downarrow \theta_{GP}$ and $\underline{\theta}^m \uparrow \theta_{GP}$ as $m \to \infty$. ∎

Lemma 5 The state $\mathbf{y}$ is uniquely absorbing and globally attractive iff $(\mu_1, \mu_2) \in \Omega_{\mathbf{y}}$ for either $\mathbf{y} = \mathbf{0}$ or $\mathbf{1}$; both $\mathbf{1}$ and $\mathbf{0}$ are absorbing iff $(\mu_1, \mu_2) \in \Omega_{\mathbf{01}}$.

Proof Without loss of generality, assume $\mu_1 \le \mu_2$ and $\delta \le 1$.
Case 1: $\mu_i \ge \frac{\lambda_i + r}{\lambda_1 + \lambda_2 + r}$ for $i = 1, 2$.
The outcome **H** can be upset when players have an incentive to deviate for a feasible path from $(y^1, y^2) = \mathbf{1}$. Because of the monotonicity of Φ^i, the incentive to deviate is the strongest if all players are anticipated to switch from H to L in the future, i.e. $y_t^i = y_0^i e^{-\lambda_i t}$ for $i = 1, 2$. Hence, the state $\mathbf{1}$ is fragile since

$$\begin{aligned} V_t^i &= (\lambda_i + r) \int_0^\infty \{y_0^j e^{-\lambda_j (t+s)} - \mu_i\} e^{-(\lambda_i + r)s} ds \\ &= y_0^j e^{-\lambda_j t} \frac{\lambda_i + r}{\lambda_1 + \lambda_2 + r} - \mu_i \le 0 \end{aligned}$$

for any t and y_0^j.
Case 2: $f_{\delta,\rho}(\mu_1) \le \mu_2 \le \frac{1+\delta\rho}{1+\delta+\delta\rho}$
Note that this implies $\mu_1 < \frac{1+\delta\rho}{1+\delta+\delta\rho}$. If $y_0^2 \le \mu_1 \frac{1+\delta+\delta\rho}{1+\delta\rho}$, then the monotonically decreasing path as in case i) works. Otherwise, consider the following path:

$$y_t^2 = e^{-\lambda_2 t}$$

and

$$y_t^1 = \begin{cases} 1 & \text{for} t < T \\ e^{-\lambda_1 (t-T)} & \text{for} t \ge T \end{cases}$$

where T satisfies

$$e^{-\lambda_2 T} = \mu_1 \frac{1+\delta+\delta\rho}{1+\delta\rho}. \tag{3.18}$$

The path described above is indeed an equilibrium since

$$V_t^1 = (\lambda_1 + r) \int_0^\infty (e^{-\lambda_2(t+s)} - \mu_1) e^{-(\lambda_1+r)s} ds$$

is greater (resp. smaller) than zero if and only if $t <$ (resp. $>$)T. On the other hand,

$$\begin{aligned} V_0^2 &= (\lambda_2 + r) \int_0^T (1-\mu_2) e^{-(\lambda_2+r)s} ds + (\lambda_2 + r) \int_T^\infty (e^{-\lambda_1(s-T)} - \mu_2) e^{-(\lambda_2+r)s} ds \\ &= f_{\delta,\rho}(\mu_1) - \mu_2 \leq 0. \end{aligned}$$

<u>Case 3</u>: $\mu_1 \leq \mu_2 \leq f_{\delta,\rho}(\mu_1)$.

Note that this also implies $\mu_1 < \frac{1+\delta\rho}{1+\delta+\delta\rho}$. First, for any feasible path, if $y_t^2 > \mu_1 \frac{1+\delta+\delta\rho}{1+\delta\rho}$, then $y_{t+s}^2 \geq y_t^2 e^{-\lambda_2 s}$, and

$$V_t^1 \geq (\lambda_1 + r) \int_0^\infty (y_t^2 e^{-\lambda_2 s} - \mu_1) e^{-(\lambda_1+r)s} ds = y_t^2 \frac{1+\delta\rho}{1+\delta+\delta\rho} - \mu_1 > 0.$$

This implies that, for $y_0^2 > \mu_1 \frac{1+\delta+\delta\rho}{1+\delta\rho}$, $V_t^1 > 0$ for all $t < T$, where T satisfies $y_0^2 e^{-\lambda_2 T} = \mu \frac{1+\delta+\delta\rho}{1+\delta\rho} < 1$. Thus,

$$y_t^1 \geq \begin{cases} 1-(1-y_0^1)e^{\lambda_1 t} & \text{if } t < T \\ (1-(1-y_0^1)e^{-\lambda_1 T})e^{-\lambda_1(t-T)} & \text{if } t > T \end{cases}$$

for all $t > 0$. Using the fact that the right hand side is continuous in y_0^1, and that $V_0^2 \geq f_{\delta,\rho}(\mu_1) - \mu_2 > 0$, there exists a neighborhood of $\mathbf{1}$ such that $V_0^i > 0$ for $i = 1, 2$, thus $\mathbf{1}$ is absorbing.

Similar arguments show all the other cases, including the case of $\delta > 1$ and the region

where the state $\mathbf{0}$ is fragile. ∎

	Stag	Hare
Stag	10, 10	0, x
Hare	x, 0	x, x

Figure 3.1: Two-Player Stag Hunt Game

.	Stag	Hare	Stag	Hare
Stag	10,10,10	0,x,0	0,0,x	0,x, x
Hare	x,0,0	x, x,0	x,0,x	x, x, x
	Stag		Hare	

Figure 3.2: Three-Player Stag Hunt Game

$$0 < x < 10$$

n	2	3		15		99	
rule		mim	med	min	med	mim	med
Limit Dynamics	5	3.33	6.67	0.67	5.33	0.10	5.05
Global Perturbation	5	3.33	6.67	0.67	5.33	0.10	5.05
Risk Dominance	5	3.82	6.18	1.34	5.26	0.34	5.01

Table 3.1: Cutoffs in Stag Hunt Game

..	H	L
H	a_1, a_2	b_1, c_2
L	c_1, b_2	d_1, d_2

$$a_i > c_i, a_i \geq d_i > b_i,\ i = 1, 2.$$

Figure 3.3: General 2×2 Game

ρ	0	0.1	0.5	1	10	10^2	∞
u	.067	..073	.097	.125	.440	.878	1.000
ℓ	.067	.053	.020	.008	.001	..000	.000
H	.067	.056	.035	.030	.053	.066	..067
L	.933	.944	.965	.970	.947	.934	.933

Table 3.2: Large Group Minimum Rule

ρ	0	0.1	0.5	1	10	10^2	∞
u	.533	.566	.673	.767	.997	1.000	1.000
ℓ	.533	.502	..398	.300	.183	.000	.000
H	.533	..536	.541	.544	.534	.533	.533
L	.467	.464	.459	.456	.466	.467	.467

Table 3.3: Large Group Median Rule

Chapter 4

Evolutionary Learning with Experimentations

4.1 Introduction

In this paper, we analyze a game played by randomly and anonymously matched players from a large population. The class of games we study are symmetric, binary action, multiperson coordination games with two strict Pareto-ranked Nash equilibria. Existing refinements are powerless to select between these equilibria. For instance, many of the stringent solution concepts proposed in the literature, such as the strategic stability of Kohlberg and Mertens [1986], are silent concerning the selection among several strict Nash equilibria. Furthermore, some recent studies on learning have also addressed the question of how a particular equilibrium will emerge in a dynamic context. Although some convergence results are obtained, these studies do not offer an equilibrium selection criterion, since in these models the strict Nash equilibria all share the same dynamic properties.

One approach for resolving equilibrium selection indeterminacy is to consider an actual adjustment process which operates in real time, and to see what limit outcomes if any might appear. We allow players to have the opportunity from time to time to revise their choices given what their opponents are currently doing, and given the correct expectation about the future play of the game—namely, perfect foresight. If this continuous revision process settles down to a unique outcome, then this outcome should be the analyst's predection of how the game might be played. Therefore, this approach has the potential to explain how equilibrium is attained, and of singling out a unique equilibrium in situations where the underlying stage game has a plethora of outcomes. Using deterministic adjustment dynamics with perfect foresight, Kim [1992a] provides a full characterization of the dynamic equilibrium outcomes as a function of the payoff matrix and an effective discount rate. More importantly and loosely speaking, an the dynamic outcome, in the limit as players become very patient, selects uniqely from the strict Nash equilibria depending on the payoff matrix. The resulting equilibrium selection suggests the following introspection arguments. Consider a player about to play a one-shot n-person coordination game. Each player faces $(n-1)$ opponents, and so there are n possibilities where in each possibility $(k-1)$ out of $(n-1)$ opponents choose action H for $k = 1, \ldots, n$. Denote A_{k-1} each of these possible events. Assume that the player places equal uniform probability $1/n$ on each event A_{k-1} for $k = 1, \ldots, n$. Notice that this probability assignment necessarily implies that a player presumes some degree of correlation between opponents' choices. Surprisingly, this outcome coincides with Carlsson and Van Damme's [1990, 1991] static notion of equilibrium selection called global perturbation.[1]

In the opposite limit as players become myopic, both strict equilibria can be simply reached, and exactly which equilibrium will be actually obtained in the long run depends crucially upon the initial state. This is reflected in the fact that, in the framework of an evolutionary process which assumes myopia, Darwinian deterministic dynamics may well possess multiple steady states and the asymptotic behavior of the system depends on the historical accident of initial conditions. Trouble persists even if we perturb the deterministic dynamic system with a one-time mutation, which is the idea behind the concept of standard evolutionary strategic stability (in brevity, ESS).[2] Moreover, we note here that the connection between myopic replicator dynamics and strategic stability or rationalizability is vacuous in coordination games, since all strict Nash equilibria simply survive strict iterative admissibility.

Another approach to resolve indeterminacy is to introduce a probabilistic flow of small mutations or experimentations, thus making the dynamic system stochastic. The resulting stochastic law of motion possesses a well-defined, steady-state

[1]Refer to Kim [1992a] for formal proof, or Section 4.5 below for summary.

[2]For an excellent survey of ESS refer to Hofbauer and Sigmund [1988].

ergodic distribution.[3] Consequently, this approach highlights certain strategy configurations as likely to be observed much more frequently than others, especially in the limit as the chance of mutations vanishes. And it turns out that the power to distinguish between multiple strict Nash equilibria returns even under myopia. The long-run state derived using stochastic evolutionary dynamics with myopia corresponds to the static equilibrium selection motivated by the following introspection arguments. Assume that each of the player's opponents choose actions H and L with probability half on each. Also assume that players do this randomization *independently* of each other. Under this assumption, a player can calculate the expected payoff from each action. The player then chooses which action to take based on this calculation.

Much of the existing literature have asserted that the limit dynamic equilibrium outcome coincides with Harsanyi and Selten's [1988] notion of risk dominance. In this paper, we provide an overview of the connection between the nature of the dynamic process and static equilibrium selection. This paper refutes the conjectured equivalence between the limit dynamic outcome and risk dominance. We also show that, only for two-person, bimatrix games, the following four equilibrium selection rules all happen to coincide: (1) deterministic dynamics with patient players, (2) stochastic evolutionary dynamics, (3) global perturbation, and (4) risk dominance. Finally, for any general pure coordination game, a much stronger result can be obtained that supports Pareto efficiency, regardless of the underlying dynamics.

Some readers might be surprised upon recognizing that the selected Nash equilibrium may differ depending on the underlying dynamics. It seems to me entirely

[3]The core mathematical idea was developed by Freidlin and Wentzell [1984] in the context of general dynamic systems, and was applied to games by Foster and Young [1990], Young [1992], Kandori, Mailath, and Rob [1992], and Kandori and Rob [1992]. However, these studies only concentrate on two-person games.

reasonable that the long-run, steady-state equilibria or social conventions which have been formed and established for a long period of time may well differ, depending on the nature of the social system, characteristics of the members consisting of the society, the type of individual interactions within one's environment, and so forth. We should not expect exactly the same prediction about behavioral pattern when we model animal actions, such as mating contests and hunting and preying contests, as when we model highly sophisticated and patient human decisions, such as choosing computer software and locating factories or stores. More importantly and interestingly enough, there exists a remarkable link between the nature of the adjustment dynamics and the selection of static equilibrium. The dynamic outcome that is obtained with more patient players seems to correspond to static behavior that assumes more correlated play by opponents, and vice versa.

The balance of the paper is organized as follows. Section 4.2 formally defines the game of interest. Section 4.3 reviews the dynamic equilibrium outcome under the deterministic adjustment dynamics with patient players. Section 4.4 analzes the long-run states under stochastic evolutionary dynamics with myopic players. Section 4.5 contains the main discussion of the ideas in this paper, such as the relationship between the nature of dynamic systems and the static equilibrium selection and the interpretations of selection criteria. The final section concludes with some comments.

4.2 The Game

We consider a symmetric n-person coordination game with binary actions, denoted by High (H) and Low (L). Consider a strategy profile in which k agents choose H with the remaining $(n-k)$ agents choosing L. For notational convenience, we denote π_k^H and π_{n-k}^L to be the payoff for a player taking H and L respectively, where $k = 1, \dots, n$. The game of interest belongs to:

$$\Omega \equiv \Big\{ \Pi \in \Re^{2n} \quad | \quad \pi^{\zeta}_{k+1} \geq \pi^{\zeta}_{k}, \forall \zeta, \forall k \text{with strict inequality for some } k; \\ \pi^{H}_{n} > \pi^{L}_{1}, \pi^{L}_{n} > \pi^{H}_{1};\ \pi^{H}_{n} \geq \pi^{L}_{n} \Big\}. \tag{4.1}$$

The first set of inequalities in Eq. (4.1) imply that a player taking a particular action is no worse off when the number of opponents taking the same action increases. The next two inequalities require that all players playing a common action be a strict Nash equilibrium. The last inequality means that the equilibrium when all players play H, denoted by **H**, is better than the one when all players play L, denoted by **L**. Now, the following preliminary result is straightforward:

Lemma 7 *If* $\Pi \in \Omega$ *then the only pure strategy equilibria of* $G(n, \Pi)$ *are two strict Nash, viz.* **H** *and* **L**.

Proof It suffices to show for all $k = 1, \dots, n-1$ both $\pi^{L}_{n-k} > \pi^{H}_{k+1}$ and $\pi^{H}_{k} > \pi^{L}_{n-k+1}$ are not satisfied, since otherwise the pure strategy profile of k players choosing H and $(n-k)$ players choosing L would be Nash. Adding the above two inequalities yields:

$$-(\pi^{L}_{n-k+1} - \pi^{L}_{n-k}) > \pi^{H}_{k+1} - \pi^{H}_{k}$$

which contradicts the definition of the Ω set. ∎

All of the Nash refinements, including the strategic stability of Kohlberg and Mertens, are powerless in selecting between these two strict Nash equilibria. Pareto efficiency is compatible with equilibrium play, so neither an incentive problem nor conflict exists. However, it is not clear whether players will be able to reach this outcome in a noncooperative situation where no direct communication is allowed. In short, strategic uncertainty matters.

4.3 Adjustment Dynamics

We begin with the following determinsitic adjustment dynamics, which was originally introduced by Matsui and Matsuyama [1991] and subsequently studied by Kim [1992a]. Time is continuous on $[0, \infty)$. The game $G(n, \Pi)$ is played repeatedly in a society with a continuum of identical players. At every point in time, each and every player is matched to form a group with $(n-1)$ other anonymous players, who are randomly drawn from the population. All players behave rationally, choosing a strategy to maximize one's expected discounted payoff. However, adjustments are costly, so that players can revise actions only periodically. More specifically, we assume that the opportunity to switch actions arrives randomly and independently across players, following a Poisson process with mean arrival rate λ. This is called an *inertia assumption* with the speed of adjustment λ. We allow players to have the chance from time to time to revise their choices given what their opponents are currently doing, and given the correct expectation about the future play of the game, namely, perfect foresight. The dynamic system is deterministic in that there is neither stochastic shocks nor aggregate uncertainty.

The strategy distribution of the society as of time t can be described by the state variable y_t, the fraction of players that are committed to action H at time t. The state space thus is $[0, 1]$. When the opportunity to switch actions arrives, players choose the action which results in higher expected discounted payoffs, with respect to the future, expected path of y, as well as their own inability to switch actions continuously. Let $\Phi(y_t)$ be the value of playing action H instead of L at time t. Denote V_t to be discounted expected payoff given perfect foresight path $y_{t+s}, s \geq 0$, i.e.,

$$V_t \equiv (\lambda + r) \int_0^\infty \Phi(y_{t+s}) e^{-(\lambda+r)s} ds \tag{4.2}$$

then players will commit to H if $V_t > 0$ or they will commit to L if $V_t < 0$, with indifference optimal if $V_t = 0$. with $r > 0$ being the discount factor. We define r/λ to be the effective discount rate or the degree of friction. To say that the friction vanishes implies that players are very patient, or that each player can revise his action whenever he wants. On the other hand, to say that the friction grows without bound implies that players only care about their immediate gains, that is players are myopic.

Chapter 3 fully characterizes the dynamic equilibrium outcome in terms of group size and effective discount rate. We need to introduce the following terminology: A state y is said to be *absorbing* if a neighborhood U of y exists such that any equilibrium path from U converges to y. It is said to be *globally attractive* if there exists an equilibrium path that reaches or converges to that state from any initial condition. It can be shown that in the limit as the friction vanishes either all players playing action H (i.e. $y = 1$) or all players playing action L (i.e. $y = 0$) will be the unique, absorbing and globally attractive state, depending upon the payoff matrix and group size. More precisely, we restate the following:

Proposition 6 *(Kim [1992])* **(a)** *In the limit as players become patient, the unique, absorbing and globally attractive state selects* **H** *if and only if:*

$$\frac{1}{n}\sum_{k=1}^{n}\pi_k^H > \frac{1}{n}\sum_{k=1}^{n}\pi_k^L \tag{4.3}$$

with **L** *selected when the inequality is reversed.* **(b)** *In the limit as players become myopic, both* **H** *and* **L** *can be selected as absorbing states.*

Surprisingly, the equilibrium selected coincides with the one selected by Carlsson and Van Damme's [1990, 1991] static notion of global perturbation. Trembles are introduced into the game in such a way that payoffs are almost but not perfectly common knowledge, and that there is a chance that each of the actions can be a dominated strategy. More precisely, each player receives a private signal about the payoffs, but is unable to fully separate the true payoff realization from one's private noisy signal. Lack of common knowledge among players makes it possible for strictly dominated strategies to exert an influence. This fact suggests that, to solve the resulting incomplete information game, we must use iterative elimination of strictly dominated strategies. The result of iterative strict dominance prescribes that all players play either H or L, depending on the payoff matrix of the original unperturbed game. Equilibrium selection based upon global perturbation refers to the one obtained at the exactly original game as common knowledge about payoffs becomes arbitrarily perfect. To recapitulate, the major argument of Chapter 3 is that the limiting dynamic outcome is equivalent to the static equilibrium selection based on global perturbation.

Proposition 6(b) suggests that in the limit as players become myopic exactly which equilibrium between **H** and **L** will be obtained in the long run depends crucially upon the initial state. In other words, no *a priori* selection among the multiple Nash equilibria can be made. How to restore some ability for equilibrium selection will be discussed in the next section.

4.4 Evolutionary Dynamics

4.4.1 Characterization

For analytical convenience, we modify the repeated societal interactions as follows. Time is discrete and is denoted by $t = \{0, 1, \ldots, \}$.[4] The game $G(n, \Pi)$ is played repeatedly in a society with a finite number of identical players N which is an integer divisible by n. The state variable z_t is the number of players adopting action H at time t, with the state space $Z = \{0, 1, \ldots, N\}$. Let $y_t = z_t/N$ be

[4] The analysis extends to a continuous time formulation, as will be mentioned later.

the fraction of players choosing action H at t. Within period t, there are a large number of random matches among the players so that each player's average payoff in that period is equal to the expected payoff. That is, the value of playing action H instead of L is equal to:

$$\Phi(z) = \sum_{k=\max\{1,n-N+z+1\}}^{\min\{z+1,n\}} \frac{\begin{pmatrix} z \\ k-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-k \end{pmatrix}}{\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}} \phi_k \tag{4.4}$$

where $\phi_k \equiv \pi_k^H - \pi_{n-k+1}^L$ is nondecreasing in k. [5]

Consider the myopic limit in the adjustment dynamics as discussed in the previous section, that is, time discount rate r becomes arbitrarily large. Together with the inertia assumption, this implies that, given the chance to move, each player adopts a best response against the current strategy configuration of the society as a whole. In other words, players commit to action H if $\Phi(z_t) > 0$, and to action L if $\Phi(z_t) < 0$. This suggests Darwinian deterministic dynamics, denoted by $\mathcal{P}(0)$, which prescribes the state $\hat{z}_{t+1} = f(z_t)$ at $t+1$ with the following properties:

(A1) $sign(f(z) - z) = sign(\Phi(z))$ for $z \neq 0$, or N.

(A2) $f(0) = 0$ and $f(N) = N$.

Notice that property (A2) says that players change their strategies only when their current strategies are strictly worse than the best ones. This assures that, once a

[5] This formula is derived by denoting z as the number of other players playing action H. The exact expression for $\Phi(z)$ is:

$$\sum_{k=1}^{\min\{z,n,N-z+1\}} \frac{\begin{pmatrix} z-1 \\ k-1 \end{pmatrix}\begin{pmatrix} N-z \\ n-k \end{pmatrix}}{\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}} \pi_k^H - \sum_{k=1}^{\min\{z+1,n,N-z\}} \frac{\begin{pmatrix} z \\ k-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-k \end{pmatrix}}{\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}} \pi_{n-k+1}^L .$$

Using this expression only increases the analytical complications without changing our results.

Nash equilibrium is reached, the society stays there forever. In particular, since the stage game $G(n, \Pi)$ has multiple strict Nash equilibria, the dynamic system $\mathcal{P}(0)$ may well possess multiple steady states and that the asymptotic behavior of the system depends on the initial condition z_0.

This problem persists even if we introduce a small, once-and-for-all disturbance into the dynamic system—this is the idea behind the concept of standard evolutionary stability. Notice that the connection between myopic replicator dynamics and strategic stability or rationalizability is vacuous with respect to coordination games, since both strict Nash equilibria simply survive strict iterative admissibility.[6]

This kind of equilibrium selection indeterminacy is resolved if we perturb the system with a constant flow of mutations or experimentations. The idea behind mutations is to test the stability of states by repeatedly subjecting them to disturbances, and observe to which states the society tends to return. Denote ε to be the mutation rate. This yields the following nonlinear stochastic dynamics:

$$z_{t+1} = \hat{z}_{t+1} + x_t^{HL} - x_t^{LH} \tag{4.5}$$

where $\hat{z}$ is the planned state prescribed by Darwinian deterministic dynamics, $x_t^{HL} \sim \text{Bin}(\hat{z}_{t+1}, \varepsilon)$ and $x^{LH} \sim \text{Bin}(N - \hat{z}_{t+1}, \varepsilon)$. The dynamic system $\mathcal{P}(\varepsilon)$ defines a $(N+1) \times (N+1)$ Markovian transition matrix, whose typical element has the following polynomial expression:

$$p_{zz'} = \sum_{k=0}^{N} \gamma_{zz'}(v)\varepsilon^v \ . \tag{4.6}$$

Given $\mathcal{P}(\varepsilon)$, the system perpetually flips around in Z. Hence, we consider a particular stochastic equilibrium concept by measuring the average proportion of time that the society will spend in each state. Formally, let Δ^N be the N

[6] Refer to Swinkels [1992] and references therein.

dimensional simplex. A *stationary distribution* or *invariant measure* is a row vector $\mu \in \Delta^N$ satisfying:

$$\mu(\varepsilon)\mathcal{P}(\varepsilon) = \mu(\varepsilon). \tag{4.7}$$

Under our assumptions, the matrix $\mathcal{P}$ is irreducible, so that the stationary distribution has certain nice properties, namely uniqueness, global stability and ergodicity.[7] Global stability implies that, independent of the initial state, the system converges to the stationary distribution μ. Ergodicity implies that μ can be interpreted as the proportion of time that the society spends in each state. We will next examine the long-run behavior of the system when the probability of mutation is arbitrarily small. To this end, we introduce the following concepts.

Definition 4.1 The *limit distribution* μ^* is defined by $\lim_{\varepsilon \to 0} \mu(\varepsilon)$.

Definition 4.2 The set of *long run equilibria* is the carrier of μ^*.

We identify the long-run states by using a graph-theoretic characterization of the invariant measures $\mu(\varepsilon)$. According to this approach, μ is a scalar multiple of a vector $q \in \Delta^N$ where:

$$q_z = \sum_{b \in B_z} \prod_{(z', z'') \in b} p_{z'z''} \tag{4.8}$$

and where B_z is the set of one particular class of directed graphs defined on the state space Z, called z-trees. A z-tree is a set of directed branches, which means that every $z' \neq z$ is the origin of exactly one branch, and that starting from any such state there is a unique sequence of branches terminating at z.

Eqns. (4.6) and (4.8) make it clear that each q_z is a polynomial expression in ε, that is:

$$q_z = \sum_{v=0}^{N} \alpha_z(v)\varepsilon^v \tag{4.9}$$

[7]Refer to any standard textbook such as Karlin and Taylor [1975].

should tend to zero as $\varepsilon \downarrow 0$. Note that the stationary distribution μ equals $q/\sum_{z=0}^{N} q_z$, and in this expression both the denominator and the numerator tend to zero as $\varepsilon \downarrow 0$. So, the identification of the long-run states, which receive positive probability in the limit, hinges on how fast each q_z vanishes. Define the cost of transition between state z and z' as $c(z, z') = |f(z) - z'|$, which is the minimum number of mutations to achieve state z' from state z. Thus, if v_z is the speed of convergence of q_z to zero, that is, $q_z = o(\varepsilon^{v_z})$, then eqns. (4.8) and (4.9) imply:

$$v_z = \min\{v | \alpha_z(v) \neq 0\} = \min_{b \in B_z} \sum_{(z'z'') \in b} c(z', z'') \,. \tag{4.10}$$

We call v_z the cost of transition to state z, and it can be thought as the difficulty of achieving state z in the long run. The problem of finding the set of long-run states is reduced to the problem of minimizing Eq. (4.10) over all states $z \in Z$ and over all trees $b \in B_z$, or we can write formally:

$$\min_{z \in Z} \min_{b \in B_z} \sum_{(z', z'') \in b} |f(z') - z''| \,. \tag{4.11}$$

In other words, the long-run states are those with the least cost of transition.

4.4.2 Equilibrium Selection

We show that either one or the other of the strict Nash equilibria of any coordination game $G(n, \Pi)$ is selected in the long run, depending on the payoff matrix and the group size. We summarize the selection criterion in the following proposition.

Proposition 7 *For N sufficiently large, the selected, unique long-run equilibrium is* **H** *if and only if:*

$$\sum_{k=1}^{n} w_k \pi_k^H > \sum_{k=1}^{n} w_k \pi_k^L \tag{4.12}$$

where the weights are defined by:

$$w_k \equiv \frac{\binom{n-1}{k-1}}{2^{n-1}} \qquad \textit{for } k = 1, \ldots, n. \tag{4.13}$$

And when the inequality is reversed, the long-run equilibrium becomes **L**.

Obviously, in the nongeneric case of equality, the selected, long-run equilibrium can be either **H** and **L**. Moreover, the limit stationary distribution places probability half on each. Two lemmas are helpful for proof of the Proposition 7.

Lemma 8 *For N sufficiently large, $\Phi(z) = 0$ has a unique root in $[0, N]$.*

Proof For $z = 0, \ldots, n-1$, eq. (4.4) can be reduced to $\Phi(z) \approx \phi_1 + \sum_{k=1}^{z} o(N^{-k})\phi_k$. Since $\phi_1 \equiv \pi_1^H - \pi_n^L < 0$ by eq. (1), the value of $\Phi(z)$ is negative if N is large enough. Similarly, for $z = N-1, N-2, \ldots, N-n+1$, the value of $\Phi(z)$ is shown to be positive if N is large enough.

Let $z = \lambda N$, where $\lambda \in (0, 1)$, then we can write for N large enough:

$$\Phi(\lambda) \approx \sum_{k=1}^{n} \binom{n-1}{k-1} \lambda^{k-1}(1-\lambda)^{n-k}\phi_k \ .$$

But since by eq. (4.1), we have:

$$\Phi'(\lambda) = (n-1)\sum_{k=0}^{n-2} \binom{n-2}{k} \lambda^{k}(1-\lambda)^{n-k-2}[(\pi_{k+2}^H - \pi_{k+1}^H) + (\pi_{n-k}^L - \pi_{n-k-1}^L)] > 0 \ .$$

Hence, $\Phi(z)$ is increasing in z, for N large enough. Combining these facts yields the desired result. ∎

Immediate from Lemma 8 is that there exists a critical population level z^* for which the two states 0 and N have basins of attraction under the dynamic $b(\cdot)$ given by $\{z < z^*\}$ and $\{z > z^*\}$, respectively. The relative sizes of these basins of attraction are a crucial determinant of the limit distribution. With respect to this, the following result is important:

Lemma 9 *For N sufficiently large and any Darwinian dynamic process $\mathcal{P}(0)$ satisfying properties (A1) and (A2), the limit distribution for $G(n, \Pi)$ puts probability one on N if $z^* < \frac{N}{2}$, or probability one on 0 when the inequality is reversed.*

Lemma 9 is directly taken from Kandori, Mailath and Rob [1991]. Although they have two-player games in mind, everything goes through with three or more players, thus the proof is omitted.

Now we are ready for the proof:

Proof of Proposition 7 In principle, we can calculate the unique root z^* as a function of n, Π and N, and then see what happens to $z^* = N/2$ as N becomes large. But this procedure is rather complicated. The trick is to plug $z = N/2$ directly into the $\Phi(z) = 0$ equation, and then see what happens in the limit as $N \to \infty$.

Without loss of generality, assume N is an even number, that is, $M = N/2$, then we have:

$$\Phi(M) = \sum_{k=1}^{n} \frac{\begin{pmatrix} M \\ k-1 \end{pmatrix} \begin{pmatrix} M-1 \\ n-k \end{pmatrix}}{\begin{pmatrix} 2M-1 \\ k-1 \end{pmatrix}} \phi_k = 0 \ . \tag{4.14}$$

The coefficient of ϕ_k is rearranged, so we have:

$$\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \frac{M(M-1)\cdots(M-(k-2))(M-1)\cdots(M-(n-k))}{(2M-1)(2M-2)\cdots(2M-(n-1))} \ .$$

This expression goes to w_k in the limit as $M \to \infty$. Plugging $\phi_k \equiv \pi_k^H - \pi_{n-k+1}^L$ and the expression for w_k into Eq. (4.14) yields the desired result. ▮

The results derived thus far are robust when extended to a continuous time formulation, as long as we maintain the assumptions that the popualtion is finite and that mutations and the opportunity to switch actions are independent over time and across players. The trick is to map continuous time into discrete time by focusing attention on the stopping times when the state changes. Even though simultaneous mutations never occur in continuous time, a sequence of single mutations can occur within a short time interval, in which no adjustment of strategy takes place. In other words, the most likely way to upset an equilibrium is to have a series of mutations within a short time interval, before the selection pressure takes place. One can show without much difficulty that the long-run equilibrium emerges irrespective of the speed of adjustment in each basin.[8]

4.5 Main Discussion

Our results for equilibrium selection come under two broad categories:

1. Equilibrium selection obtained under deterministic perfect foresight adjustment dynamics with patient players (Proposition 6).

2. Equilibrium selection obtained under stochastic evolutionary dynamics with myopic players (Proposition 7).

I now would like to address the issue of how these two categories differ.

The assumption of inertia or costly adjustment is common to both. Given the chance to move, players choose a best response with respect to some suitably defined objective function, in our two cases, the expected discounted payoff calculated under perfect foresight and the average payoff given the current strategy configuration. Neither the time formulation nor population size matter. Stochastic shocks—through the possibility of mutations—in the dynamics with myopia plays a crucial role in reviving the power to select equilibria, but not in its characterization. The crucial difference is how players value the future, namely patience. Let us refer to P-selection as equilibrium selection according to eq. (4.3) of Proposition 6, and M-selection as equilibrium selection according to eqns (4.12) and (4.13) of Proposition 7; here, "P" stands for patience and "M" stands for myopia.

[8]Refer to Kandori [1991] for details.

Now, consider the relationship between the long-run equilibrium obtained under dynamics in actual time to the static, one-shot equilibrium selected by introspection arguments. Consider the situation in which a player about to play the one-shot game $G(n, \Pi)$. The player is confronted with $(n-1)$ opponents, meaning that there are n possible events A_{k-1}, for $k = 1, \ldots, n$, in which $(k-1)$ out of $(n-1)$ opponents choose action H. For this game, M-selection is interpreted as follows: the player assumes that each opponent chooses action H and L with probability half on each. The player further assumes that all the opponents do this randomization independently of each other. Under these assumptions, it can be easily checked that the event A_{k-1} occurs with probability w_k defined by eq. (4.13). Hence, the player should choose either actions H or L if the expected payoff from either is greater, as it is in eq. (4.12). On the other hand, P-selection directly places equal probability $1/n$ on each possible event A_{k-1}. This probability assignment necessarily implies that the player presumes some correlation among the opponents' choices, contrary to the independence assumption of the player's opponents' choices in M-selection. If the group size is $n = 2$, the player has only one opponent, so that this distinction simply disappears. Hence, the two equilibrium selection categories coincide with each other, and by chance, to risk dominance.

Since in the case of a 2×2 game, the player has only one opponent, so that there is no distinction between the two selection categories, we have that M- and P-selection coincide with each other. Furthermore, we have the following proposition:

Proposition 8 *If and only if $n = 2$, the following are equivalent:*

(a) M-selection,

(b) P-selection,

(c) global perturbation,

(c) risk dominance.

Proof The preceding arguments in the text above imply that (a) and (b) coincide when $n = 2$. Equivalence between (b) and (c) is true in general Chapter 4.3 of this paper. Carlsson and Van Damme [1990] verify the equivalence between (c) and (d) for two-person, bimatrix games. ∎

The limit behavior of Blume's [1991] dynamic process with respect to parametric changes making strategy revisions a best response is shown to give rise to the same outcome as risk dominance in 2×2 coordination games. Kandori, Mailath, and Rob [1992]—upon which our Section 4 depends heavily—and Fudenberg and Harris [1992] show that, in 2×2 games, as the mutation rate and noise go to zero, the ergodic distribution becomes concentrated on the risk dominant equilibrium.[9] Lastly, Matsui and Matsuyama [1991]—from which our model borrows heavily—shows an equivalence between risk dominance and dynamic stability in 2×2 games of common interest. From the viewpoint of Proposition 8 above, the first three papers (i.e. Blume, KMR and FH) claim nothing but the equivalence between M-selection and risk dominance, with the last paper (ie, MM) claiming the equivalence between P-selection and risk dominance.

We now study a generalized pure-coordination or simply voting game, where the payoff to the player playing action $s = 1, 2, \ldots, m$ is described as follows:

$$\pi^s(\sharp(1), ..., \sharp(m)) = \begin{cases} a_s & \text{if } \sharp(s) \geq \kappa \\ 0 & \text{otherwise} \end{cases}$$

where $\sharp(s)$ denotes the total number of players choosing action s, and κ may be $2, .., n$. Moerover, all coordinated equilibria are ordered, that is, $0 \leq a_s \leq a_{s'}$, $\forall s <$

[9]Fudenberg and Harris needs to be read with some care, since in this paper, the population is large and the random perturbation is introduced by a Brownian motion, so that the stochastic shocks are necessarily correlated across players. This is in sharp contrast to our assumption concerning the independence of the random shocks across players and over time.

s'. This class of games, denoted $G(n, m; \Pi^\kappa)$ to emphasize the importance of κ. possesses m pure strategy Pareto rankable Nash equilibria, everyone's choosing action $s = 1, 2, ..., m$. It requires that both the voting rule (represented by κ) and the security (normalized to zero) be identical over all choices. Then we have:[10]

Proposition 9 *In any* $G(n, m; \Pi^\kappa)$, *the Pareto efficient Nash equilibrium is supported, regardless of the underlying dynamics.*

Proof is lengthy, but the idea is intuitive. The previous sections suggest that Pareto efficiency is guaranteed when the number of actions is two, i.e. $m = 2$. With three or more actions, we apply the selection criterion in a pairwise way. The only case that we have to worry about is lack of transitivities, but this cannot occur in the present class of games.

proof

(1) Deterministic Adjustment Dynamic with Patient Players:

All the proofs of Section 3 apply straightforwardly, so we omit them. After all, we are able to show that: if $\rho \in (0, \bar{\rho}]$, then the Pareto efficient outcome is uniquely absorbing and globally attractive and thus is robust with respect to global perturbation.

(2) Stochastic Evolutionary Dynamics:

Given a chance to move and the state $\mathbf{z} = (z^1, ..., z^m)$, the expected average payoff for the player who has been choosing action s is calculated as

$$\begin{cases} f_\kappa(z^s - 1)a_s & \text{if he chooses } s \text{ again} \\ f_\kappa(z^{s'})a_{s'} & \text{if he chooses } s' \neq s \end{cases} \tag{4.15}$$

[10] We can easily construct counterexamples demonstrating the fact that both identical rule and equal security are necessary and sufficient to guarantee the Pareto efficiency.

where

$$f_\kappa(z) = \sum_{k=\max\{1,n-N+z+1\}}^{\min\{z+1,n\}} \frac{\begin{pmatrix} z \\ k-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-k \end{pmatrix}}{\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}},\ z \in Z \equiv \{0,1,...,N\} \tag{4.16}$$

The next lemma is just a technical result but plays an important role in what follows.

Lemma 10 *For any κ, the function $f_\kappa(z)$ is strictly increasing in $z \in Z$.*

Proof Via mathematical induction. Without loss of generality, assume $z \in \{n-1, n, ..., N-n\}$. For $\kappa = n$, a direct calculation shows that

$$f_n(z) - f_n(z-1) = \begin{pmatrix} z-1 \\ n-2 \end{pmatrix}\begin{pmatrix} N-z-1 \\ 0 \end{pmatrix} > 0.$$

Now suppose it is true for $\kappa + 1$ that

$$f_{\kappa+1}(z) - f_{\kappa+1}(z-1) = \begin{pmatrix} z-1 \\ \kappa-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-\kappa-1 \end{pmatrix},$$

then

$$\begin{aligned} & f_\kappa(z) - f_\kappa(z-1) \\ = \ & [f_{\kappa+1}(z) - f_{\kappa+1}(z-1)] + [\begin{pmatrix} z \\ \kappa-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-\kappa \end{pmatrix} - \begin{pmatrix} z-1 \\ \kappa-1 \end{pmatrix}\begin{pmatrix} N-z \\ n-\kappa \end{pmatrix}] \\ = \ & \begin{pmatrix} z \\ \kappa-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-\kappa \end{pmatrix} - [\begin{pmatrix} z-1 \\ \kappa-1 \end{pmatrix}\begin{pmatrix} N-z \\ n-\kappa \end{pmatrix} - \begin{pmatrix} z-1 \\ \kappa-1 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-\kappa-1 \end{pmatrix}] \\ = \ & \begin{pmatrix} z-1 \\ \kappa-2 \end{pmatrix}\begin{pmatrix} N-z-1 \\ n-\kappa \end{pmatrix} > 0. \end{aligned}$$

∎

Lemma 11 *Any mixed strategy is unstable.*

Proof Assume not, i.e. there exist $s, s' \in C(\mathbf{z})$ with $s < s'$, and both s and s' are best reponses to $\mathbf{z}$. Then we get

$$f(z^{s'} - 1)a_{s'} \geq f(z^s)a_s > f(z^s - 1)a_s \geq f(z^{s'})a_{s'} > f(z^{s'} - 1)a_{s'}.$$

The strict inequalities follow from Lemma 7 and the weak inequalities follow from the presumed optimality of s and s' relative to $\mathbf{z}$. The contracdiction establishes the desired result. ∎

Lemma 12 *The collection of limit sets is* $\{e^s\}_{s=1}^m$.

This last lemma states that the Darwinian dynamic for the game $G(n, m; \Pi^\kappa)$ converges to a pure strategy Nash configuration with probability one. The same logic as in Proposition 7(2) of KR applies, so proof is omitted. Now we are ready to verify the Pareto efficiency under stochastic evolutionary dynamics:

Recalling Eq. (16), the first task is to compute costs of transition $C_{s's}$ between limit sets, e^s and $e^{s'}$. Assume the society is initially clustered at $e^{s'}$, then the minimum number of mutations, x, needed to switch it over into the basin of attraction of e^s is determined by $f(x)a_s \geq f(N-1-x)a_{s'}$. This represents an immediate jump to escape the best response region of s', and the triangular inequality argument of KR's Proposition 5 guarantees that no gradual escape is less costly than this immediate jump. Note that we mutate individuals taking s' into s, because any other mutation will only raise the transition cost more. Thus, the cost of transition $C_{s's}$ is the minimum integer x satisfying

$$f(x) \geq f(N-1-x)\frac{a_{s'}}{a_s} \tag{4.17}$$

It has a unique root, since Lemma 7 implies that the left hand side of Eq. (31) is strictly increasing and so its right hand side is strictly decreasing in x.

Since a pure coordination game $G(n, m; \Pi^\kappa)$ specifies $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m$, we can easily check that

$$C_{s'm} < C_{s's},\ \forall s < m, \forall s' \neq s,\ \text{and}\ C_{m,m-1} < C_{s',m-1},\ \forall s' < m-1.$$

Therefore, the first step of the optimum branching algorithm as in Appendix B of KR is to choose a minimum cost outgoing branch from each state, which results in the system of branches $(s \to m), \; s = 1, 2, ..., m-1,$ and $(m \to m-1)$. The longest branch among these is of length $C_{m,m-1}$. Therefore we drop it, and are left with an m-tree. This completes the algorithm. ∎

Corollary 4 *In any pure coordination game, the dynamic equilibrium outcome selects the Pareto-efficient outcome* **H***, irrespective of the details of the underlying adjustment dynamics.*

4.6 Final Remarks

We have generalized results on equilibrium selection in the direction of group size. However, the assumption of binary strategies is obviously restrictive. A full characterization of the long-run states for broader classes of games beyond those studied in this paper seems difficult, because of the numerous case distinctions and complicated calculations. This is the reason why Kandori and Rob [1992] ends up only solving the two-player, three-action supermodular game. Yet, it does seem valuable to characterize the general properties of the long-run states for broader classes of multiperson games, such as multiperson supermodular games. We will have to await further research in this direction for answers.

Chapter 5

Reference

1. L. Anderlini and H. Sabourian, Cooperation and Effective Computability, mimeo, University of Cambridge, 1991.

2. R.J. Aumann and S. Sorin, Cooperation and Bounded Recall, *Games and Economic Behavior* 1 (1989), 5-39.

3. K. Binmore and L. Samuelson, Evolutionary Games in Repeated Games Played by Finite Automata, *Journal of Economic Theory* 57 (1992), 278-305.

4. L. Blume, The Statistical Mechanics of Strategic Interactions, mimeo, Cornell University, 1991.

5. L. Blume and D. Easley, Evolution and Market Behavior, *Journal of Economic Theory* (1992), 9-40.

6. R. T. Boylan, Continuous Approximation of Dynamical Systems with Randomly Matched Individuals, mimeo, Washington University, 1992.

7. Bulow, J., and K. Rogoff, Sovereign Debt: Is to Forgive to Forget ?, *American Economic Review* 79 (1989a), 43-50.

8. ——————(1989b), A Constant Recontracting Model of Sovereign Debt, *Journal of Political Economy* 97, 155-78.

9. Camerer, C., and K. Weigelt, Experimental Tests of a Sequential Equilibrium Reputation Model, *Econometrica* 56 (1988), 1-36.

10. D. Canning, Average Behavior in Learning Models, *Journal of Economic Theory,* 57 (1992), 442-72.

11. H. Carlsson and E. van Damme, Global Games and Equilibrium Selection, CenTER # 9052, Tilburg University, 1990.

12. ———————————- , Equilibrium Selection in Stag Hunt Games, CenTER # 9170, Tilburg University, 1991.

13. R. W. Cooper, D. V. DeJong, R. Forsythe, and T. W. Ross, Selection Criteria in Coordination Games: Some Experimental Results, *American Economic Review* 80 (1990), 218-33.

14. ———————————- , Communication in Coordination Games, *Quarterly Journal of Economics* 107 (1992), 739-71.

15. V. Crawford, An Evolutionary Interpretation of Van Huyck, Battalio, and Beil's experimental results on coordination games, *Games and Economic Behavior* 3 (1991), 25-59.

16. D. Diamond, Reputation Acquisition in Debt Markets, *Journal of Political Economy* (1989), 828-62.

17. G. Ellison, Learning, Local Interaction, and Coordination, mimeo, MIT, 1991.

18. G. Ellison and D. Fudenberg, Rules of Thumb for Social Learning, mimeo, MIT, 1992.

19. G.W. Evans and R. Guesnerie, Rationalizability, Strong Rationality, and Expectational Stability, mimeo, London School of Economics, 1992.

20. D. Foster and P. Young, Stochastic Evolutionary Game Dynamics," *Theoretical Population Biology* 38 (1990), 219-32.

21. Freidlin, M. I., and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, 1984.

22. D. Fudenberg, Explaining Cooperation and Commitment in Repeated Games, in J.-J. Laffont (eds.), *Advances in Economic Theory Vol.I: Sixth World Congress of the Econometric Society,* Cambridge University Press, 1992.

23. D. Fudenberg and C. Harris, Evolutionary Dynamics with Aggregate Shocks, *Journal of Economic Theory,* 57 (1992), 420-41.

24. D. Fudenberg and D. Kreps, Reputation in the Simultaneous Play of Multiple Opponents, *Review of Economic Studies* (1987), 541-68.

25. ——————————-, *Lecture on Learning and Equilibrium in Strategic Form Games,* CORE Lecture Series, Belgium, 1992a.

26. ——————————-, Learning Mixed Equilibria, mimeo, Stanford Graduate School of Business and MIT, 1992b.

27. D. Fudenberg, D. Kreps and D. K. Levine, On the Robustness of Equilibrium Refinements, *J. Economic Theory* 48 (1988), 261-79.

28. D. Fudenberg, D. Kreps, and E. Maskin, Repeated Games with Long and Short run Players, *Review Economic Studies* (1990), 555-73.

29. D. Fudenberg and D. K. Levine, Reputation and Equilibrium Selection in Games with a Patient Player, *Econometrica* 57 (1989a), 759-78.

30. ——————————-, Efficiency and Observability with Long-run and Short-run Players, mimeo, MIT and UCLA, 1989b.

31. ——————————-, Reputation when Strategies Are Unobserved, *Review Economic Studies* (1992a).

32. ——————————————-, Steady State Learning and Nash Equilibrium, mimeo, MIT and UCLA, 1992b.

33. ——————————————-, Self-Confirming Equilibria, mimeo, MIT and UCLA, 1992c.

34. I. Gilboa and A. Matsui, Social Stability and Equilibrium, *Econometrica* 59 (1991), 859-67.

35. N.S. Glance and B.A. Huberman, Dynamics of Social Dilemmas, mimeo, Xerox Palo Alto Research Center, California, 1992.

36. E. Green, Individual Level Randomness in a Nonatomic Population, mimeo, University of Pittsburgh, 1989.

37. S. Grossman, The Informational Role of Warranties and Private Disclosure about Product Quality, *Journal of Law and Economics* (1981), 461-83.

38. G. W. Harrison and J. Hirshleifer, An Experimental Evaluation of Weakest Link/Best Shot Models of Public Goods, *J. Political Economy* 97 (1989), 201-25.

39. J. C. Harsanyi, Games with Randomly Disturbed Payoffs: A New Rationale for Mixed Strategy Equilibrium Points, *International Journal of Game Theory* 2 (1973), 1-23.

40. J. C. Harsanyi and R. Selten, *A General Theory of Equilibrium Selection in Games,* MIT Press, Massachussetts, 1988.

41. O. Hart and J. Tirole, Contract Renegotiation and Coasian Dynamics, *Review of Economic Studies* (1988), 509-40.

42. HOFBAUER, J., AND K. SIGMUND, THE THEORY OF EVOLUTION AND DYNAMICAL SYSTEMS, Cambridge University Press, London, 1988.

43. J. JORDAN, Bayesian Learning in Normal Form Games, *Games and Economic Behavior* 3 (1991), 60-81.

44. E. KALAI AND E. LEHRER, Rational Learning Leads to Nash Equilibrium, mimeo, Northwertern University, 1991.

45. M. KANDORI, Independent Mutations and Long Run Equilibria in Games: A Continuous Time Formulation," mimeo, Princeton University, 1991.

46. ————, Social Norms and Community Enforcement, *Review of Economic Studies* 59 (1992), 63-80.

47. M. KANDORI, G. MAILATH, AND R. ROB, Learning, Mutation and Long Equilibria in Games, forthcoming *Econometrica,* 1992.

48. M. KANDORI AND R. ROB, Evolution of Equilibria in the Long Run: A General Theory and Applications, mimeo, Princeton University, 1992.

49. S. KARLIN AND H. M. TAYLOR, *A First Course in Stochastic Processes,* Academic Press, San Diego, 1975.

50. Y. KIM, Characterization and Properties of Reputation Effects in Extensive Form Games, mimeo, UCLA, 1990.

51. ————, Adjustment Dynamics with Patient Players in Coordination Games, working paper #667, UCLA, 1992a.

52. ————, Adjustment, Evolution, and Equilibrium Selection in Games, mimeo, UCLA, 1992b.

53. B. Klein and K. Leffler, The Role of Market Forces in Assuring Contractual Performance, *Journal of Political Economy* 89 (1981), 615-41.

54. E. Kohlberg and J.-F. Mertens, On the Strategic Stability of Equilibria, *Econometrica* 54 (1986), 1003-38.

55. D. Kreps, *Game Theory and Economic Modelling,* Clarendon Lectures in Economics, Oxford University Press, 1990.

56. D. Kreps and R. Wilson, Sequential Equilibrium, *Econometrica* 50 (1982a), 863-94.

57. ——————————, Reputation and Imperfect Information, *Journal of Economic Theory* 27 (1982b), 253-79.

58. V. Krishna, Games with Strategic Complementaritiies, mimeo, Harvard Business School, 1992.

59. P. Krugman, *Geography and Trade,* MIT Press, Massachussetts, 1991.

60. G.J. Mailath, Introduction: Symposium on Evolutionary Game Theory, *Journal of Economic Theory* 57 (1992), 259-77.

61. A. Matsui, Cheap Talk and Cooperation in Society, *Journal of Economic Theory* 54 (1991), 245-58.

62. ————, Best Response Dynamics and Socially Stable Strategies, *J. Economic Theory* 57 (1992), 343-62.

63. A. Matsui and K. Matsuyama, An Approach to Equilibrium Selection, Hoover Institution Working Paper E-91-14, Stanford University, 1991.

64. A. Matsui and R. Rob, Evolution, Rationality, and Equilibrium Selection in Societal Games, mimeo, University of Pennsylvania, 1991.

65. P. Milgrom and J. Roberts, Predation, Reputation, and Entry Deterrence, *Journal of Economic Theory* 27 (1982), 280-312.

66. ——————————, Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities, *Econometrica* 58 (1990), 1255-77.

67. ——————————, Adaptive and Sophisticated Learning in Repeated Normal Form Games, *Games and Economic Behavior* 3 (1991), 82-100.

68. M.A. Nowak and R.M. May, Evolutionary Games and Spatial Chaos, *Nature* 359 (October 1992), 826-9.

69. A. Rubinstein, The Electronic Mail Games: Strategic Behavior under "Almost Common Knowledge," *American Economic Review* 79 (1989), 385-91.

70. L. Samuelson, How to Tremble if You Must, mimeo, University of Wisconsin, 1991.

71. T. Schelling, *The Strategy of Conflict*, Harvard University Press, Massachussetts, 1960.

72. K.M. Schmidt, Commitment through Incomplete Information in a Simple Repeated Bargaining Model, mimeo, 1990.

73. ——————, Reputation in Games with Conflicting Interest, forthcoming in *Journal of Economic Theory,* 1992.

74. R. Selten, The Chain Store Paradox, *Theory and Decision* 9 (1977), 127-59.

75. L.S. Shapley, Some Topics in Two-person Games, *Annals of Mathematics Studies* 5 (1964), 1-28.

76. J. Swinkels, Evolutionary Stability With Equilibrium Entrants, *J. Economic Theory* 57 (1992a), 306-32.

77. ——————, Adjustment Dynamics and Rational Play in Games, mimeo, Stanford University, 1992b.

78. H.P. Young, The Evolution of Conventions, forthcoming in *Econometrica,* (1992).

79. H.P. Young and D. Foster, Cooperation in the Short and in the Long run, *Games and Economic Behavior* 3 (1991), 145-56.

80. J. B. van Huyck, R. C. Battalio, and R. O. Beil, Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure, *American Economic Review* 80 (1990), 234-48.

81. ——————————————-, Strategic Uncertainty, Equilibrium Selection, and Coordination Failure in Average Opinion Games, *Quarterly J. Economics* 106 (1991), 885–910.